PERGAMON

International Journal of Solids and Structures 40 (2003) 1713–1744

INTERNATIONAL JOURNAL OF SOLIDS and STRUCTURES

www.elsevier.com/locate/ijsolstr

Extrema of Young's modulus for cubic and transversely isotropic solids ☆

Antonio Cazzani *, Marco Rovati [1]

Dipartimento di Ingegneria Meccanica e Strutturale, Università degli Studi di Trento, Via Mesiano 77, I–38050 Povo di Trento, Italy

Received 28 February 2002; received in revised form 19 November 2002

Abstract

For a homogeneous anisotropic and linearly elastic solid, the general expression of Young's modulus $E(\boldsymbol{n})$, embracing all classes that characterize the anisotropy, is given. A constrained extremum problem is then formulated for the evaluation of those directions $\boldsymbol{n}$ at which $E(\boldsymbol{n})$ attains stationary values. Cubic and transversely isotropic symmetry classes are dealt with, and explicit solutions for such directions $\boldsymbol{n}$ are provided. For each case, relevant properties of these directions and corresponding values of the modulus are discussed as well. Results are shown in terms of suitable combinations of elements of the elastic tensor that embody the discrepancy from isotropy. On the basis of such material parameters, for cubic symmetry two classes of behavior can be distinguished and, in the case of transversely isotropic solids, the classes are found to be four. For both symmetries and for each class of behavior, some examples for real materials are shown and graphical representations of the dependence of Young's modulus on direction $\boldsymbol{n}$ are given as well.

Keywords: Anisotropic elasticity; Young's modulus; Cubic symmetry; Hexagonal symmetry

1. Introduction

The macroscopic mechanical behavior of a solid is strongly conditioned by its microstructural properties. For most macroscopically isotropic materials which are commonly employed in engineering practice, it is enough to specify only two elastic coefficients in order to give a complete description of the material. However, many materials cannot be considered as isotropic; among these (at the microscale level) crystals and polycrystals, the latter constituted by grains individually anisotropic, or (at macroscale level) composites and fiber reinforced materials. Indeed, many man-made and naturally occurring substances appear as aggregates of crystals, or polycrystals, with non-random distribution of orientations (texture). In such

☆ Friendly dedicated to Professor Roberto Contro on the occasion of his 60th birthday.

* Corresponding author. Tel.: +39-461-882522; fax: +39-461-882599.

E-mail addresses: antonio.cazzani@ing.unitn.it (A. Cazzani), marco.rovati@ing.unitn.it (M. Rovati).

[1] Tel.: +39-461-882523.

0020-7683/02/$ - see front matter © 2002 Elsevier Science Ltd. All rights reserved.
doi:10.1016/S0020-7683(02)00668-6

textured polycrystals, also the macroscopic properties are anisotropic, i.e., directionally dependent. In particular, the anisotropy properties in a polycrystal aggregate depend both on the texture of the polycrystal itself and also on the inherent anisotropy of the single-crystal.

The anisotropy of a physical property is generally restricted by certain symmetry considerations, which partly follow from the symmetry elements of the underlying material structure. Symmetry considerations are in fact of paramount concern in the treatment of the directionality of material properties.

A basic structural element, common to most materials, is the crystal structure. Thus, the basic form of structural symmetry in an aggregate is that contained in the crystal structure.

The effects of crystal symmetry are exhaustively covered in Nye (1957) and in Ting (1996).

As far as elasticity is concerned, in anisotropic solids it is then necessary to specify all the independent elements of the elastic tensor $\mathbb{C}$, whose number can range from 3 (cubic system) to 21 (triclinic system), in dependence on the elastic symmetries that the solid exhibits (Gurtin, 1972). The macroscopic behavior of a solid is then strongly related to its anisotropic properties, which can reveal, in some materials, an anisotropy degree decidedly non-negligible and in some cases so extreme to suggest the proximity of material instability.

Single crystals are classified into 32 classes, on the basis of homogeneous physical properties; it has been shown (Smith and Rivlin, 1958) that these classes reveal 11 (plus isotropy) different types of elastic energy, often indicated with the notations $\mathscr{C}_1, \mathscr{C}_2, \ldots, \mathscr{C}_{11}$ (Coleman and Noll, 1964). Then, in linear elasticity, the 11 types of elastic energy reduce to 9 (Huo and Del Piero, 1991) and the relevant anisotropic symmetry classes to 8, including isotropy (Forte and Vianello, 1996; Chadwick et al., 2001). On the basis of such classification it is spontaneous to evaluate the behavior of the engineering elastic constants for the various elastic symmetries.

In the present work, the directional dependence of Young's modulus is studied with reference to the strongest elastic symmetries, namely the cubic and transversely isotropic (hexagonal) symmetries. Pioneering works in this direction are those by Goens (1933) and Schmid and Boas (1935), where some two-dimensional representations of the directional dependence of the Young's modulus and of the shear modulus, mostly based on experimental investigations, are given. In particular, Schmid and Boas (1935) from some tests on real materials provide effective pictures of plaster models for the three-dimensional surface generated by the Young's modulus.

A few planar analytically deduced drawings similar to the previous ones are also presented by Wooster (1949). The results are also referenced in classical textbooks (e.g., Nye, 1957). The case of cubic symmetry has been more recently re-discussed by Hayes and Shuvalov (1988), and some features about the general behavior of cubic solids are also investigated in Boulanger and Hayes (1995).

The aim of this paper is to theoretically investigate the elastic response of anisotropic solids (cubic and transversely isotropic ones, in particular), to analytically deduce a rational classification in terms of Young's modulus, and to recognize the appearance of such categories among real materials. This approach could be usefully applied for controlling the mechanical response of man-made materials when stiffness requirements are of concern.

The problem is formulated, in the most general form, in Section 2 and dealt with as a constrained extremum problem for the determination of those directions $\boldsymbol{n}$ along which Young's modulus attains its stationary values. The formulation is general and expresses $E(\boldsymbol{n})$ as a function of the components of the unit vector $\boldsymbol{n}$, that labels a direction, and of the Cartesian components of the fourth-order elastic tensor. The problem is equivalent to that, formulated by Ostrowska-Maciejewska and Rychlewski (2001), of finding the extrema of the stored elastic energy density for a solid under uniaxial tension.

In Section 3, the problem is then specialized to the case of cubic symmetry, where two different categories of behavior are pointed out, each characterized by the sign of a material parameter β_1, responsible of the degree of anisotropy. For practical reasons and accepted custom, in what follows, the usual Voigt's contracted representation of stress, strain and elastic tensors is adopted (Gurtin, 1972). Although this notation

provides a vector and matrix representation of the tensors which is physically meaningful, it shows, nonetheless, a loss of tensorial character. An alternative formulation (Walpole, 1984) is also mentioned, leading to the same results, but the Voigt's choice is preferred on the basis of its well-established use in current literature (Nye, 1957; Hearmon, 1961; Fedorov, 1968; Lekhnitskii, 1981; Ting, 1996). For the cubic case, the sign of the material parameter β_1 (or the value of an analogous, but dimensionless, parameter B_{cub}) fixes the directions along which the Young's modulus attains its absolute maxima and minima. It is shown that these directions can be aligned either with the edges of a cube or along its diagonals. Directions of relative extrema of $E(\boldsymbol{n})$ do exist as well and are here characterized. Several examples of real materials, compounds and alloys corresponding to positive and negative values of material parameter β_1 are then given. Among these real behaviors, the effect on Young's modulus of some extreme anisotropy degrees is also shown.

Hexagonal symmetry, and in particular that of a transversely isotropic solid, is then considered in Section 4. In this case two material parameters, α_2 and β_2 (or their dimensionless counterparts A_{hex} and B_{hex}), are defined and adopted as a measure of the level of anisotropy. In this way, the sign of α_2 along with that of the difference $(\beta_2 - \alpha_2)$ allow to distinguish the solutions of the problem in four categories of behavior. For each of such classes, it is shown that Young's modulus attains its extremum values either on the isotropy plane, or on the isotropy axis or again on the generic parallel of the surface (of revolution) generated by $E(\boldsymbol{n})$. Moreover, it is shown that the obtained solutions must be restricted by suitable feasibility conditions on the elastic constants, in order to maintain physical sense. Finally, also for transversely isotropic solids, all the treated cases are related with real materials, i.e., compounds or alloys that satisfies the previously obtained conditions. The directional dependence of Young's modulus is then shown by means of suitable graphical representations.

Information and data for anisotropic real materials can be easily found in literature, both in classical texts and tables (Huntington, 1958; Edington, 1974) and in more recent treatises (Kocks et al., 2000; Levy et al., 2001). Data for the examples presented in this work are checked with these references, but are principally taken from the largest source of information available (Landolt and Börnstein, 1992).

2. Problem formulation

A linearly elastic, homogeneous and anisotropic solid, with positive definite stored energy, is considered. The anisotropic elastic character of the material is obviously reflected on Young's modulus E, which is therefore a function of direction in the solid. In the following, for any unit vector $\boldsymbol{n}$, the modulus will be expressed as the function $E = E(\boldsymbol{n})$. The problem considered here consists in the evaluation of the behavior of that function and, in particular, in the explicit computation of those directions $\boldsymbol{n}$ for which $E(\boldsymbol{n})$ is stationary, followed by a discussion of these critical points.

The type of elastic anisotropy of the material, that is the symmetry group to which it belongs (Gurtin, 1972), is reflected on the form of the fourth-order elasticity tensor $\mathbb{C}$, here always supposed to be positive definite.

In order to obtain the explicit expression of Young's modulus as a function of the direction, the continuum is subjected to a unit dipole acting in the direction defined by the unit vector $\boldsymbol{n}$. The stress field corresponding to the unit dipole, in absence of body forces, is given by:

$$\boldsymbol{\sigma} = \boldsymbol{n} \otimes \boldsymbol{n} \tag{1}$$

and is associated, through the elasticity law, to the strain field:

$$\boldsymbol{\epsilon} = \mathbb{S}[\boldsymbol{\sigma}] = \mathbb{S}[\boldsymbol{n} \otimes \boldsymbol{n}]. \tag{2}$$

In Eq. (2) $\mathbb{S} = \mathbb{C}^{-1}$ indicates the positive definite fourth-order compliance tensor.

In view of characterizing the relationship which links the stress and the strain fields, in the direction $\boldsymbol{n}$, the strain tensor (2) is projected along that direction. The expression which defines Young's modulus as a function of direction $\boldsymbol{n}$ follows immediately:

$$\epsilon(\boldsymbol{n}) = \frac{1}{E(\boldsymbol{n})} = \boldsymbol{n} \otimes \boldsymbol{n} \cdot \mathbb{S}[\boldsymbol{n} \otimes \boldsymbol{n}]. \tag{3}$$

If a Cartesian orthogonal reference frame $Ox_1x_2x_3$ is considered, then expression (3) can be written in index form as:

$$\frac{1}{E(\boldsymbol{n})} = S_{ijhk} n_i n_j n_h n_k, \tag{4}$$

where indices i, j, h, k range from 1 to 3, and the usual rule of sum over a repeated subscript is assumed. Notice that the expression of $1/E(\boldsymbol{n})$ corresponds to that of the stored elastic energy (see Ostrowska-Maciejewska and Rychlewski, 2001) when, as in the present case, a uniaxial stress state of unit modulus is considered. This allows for the interpretation of the directions corresponding to extrema of Young's modulus as directions of maxima and minima for the stored energy function.

In (4) the S_{ijhk} values represent the Cartesian components of the elastic compliance tensor $\mathbb{S}$ in the given reference frame, and are subjected to some usual restrictions. First, they must be subordinate to the minor symmetries $S_{ijhk} = S_{jihk} = S_{ijkh}$ resulting from the symmetry of the stress and strain tensors. Secondly, also the major symmetry $S_{ijhk} = S_{hkij}$ holds, as a consequence of the requirement that no work be produced by an elastic material in a closed loading cycle.

In order to evaluate the direction $\boldsymbol{n}$ for which the modulus $E(\boldsymbol{n})$—or its reciprocal $1/E(\boldsymbol{n})$—attains extreme values, the following Lagrangian function is introduced:

$$\mathscr{L}(\boldsymbol{n}, \lambda) = \boldsymbol{n} \otimes \boldsymbol{n} \cdot \mathbb{S}[\boldsymbol{n} \otimes \boldsymbol{n}] + \lambda(\boldsymbol{n} \cdot \boldsymbol{n} - 1), \tag{5}$$

where λ is a Lagrangian multiplier associated to the constraint $\boldsymbol{n} \cdot \boldsymbol{n} = 1$ over the unit vector $\boldsymbol{n}$. In terms of components, the Lagrangian function is then written as:

$$\mathscr{L}(n_i, \lambda) = S_{ijhk} n_i n_j n_h n_k + \lambda(n_i n_i - 1). \tag{6}$$

The stationarity conditions for the Lagrangian function $\mathscr{L}$ are thus:

$$\begin{cases} \dfrac{\partial \mathscr{L}(\boldsymbol{n}, \lambda)}{\partial \boldsymbol{n}} = 0 \\ \dfrac{\partial \mathscr{L}(\boldsymbol{n}, \lambda)}{\partial \lambda} = 0 \end{cases} \quad \text{or, in index notation} \quad \begin{cases} \dfrac{\partial \mathscr{L}(n_i, \lambda)}{\partial n_j} = 0 \\ \dfrac{\partial \mathscr{L}(n_i, \lambda)}{\partial \lambda} = 0 \end{cases} \tag{7}$$

and can be explicitly written, making use of the symmetries on $\mathbb{S}$, as:

$$\begin{cases} 2S_{ijhk} n_j n_h n_k + \lambda n_i = 0 \\ n_i n_i - 1 = 0. \end{cases} \tag{8}$$

In view of more tractable computations, a change of notation is now performed. The need to evaluate the three-dimensional Hooke's law, i.e., $\epsilon_{ij} = S_{ijhk}\sigma_{hk}$, in a convenient fashion, results in a number of notational conventions (Mehrabadi and Cowin, 1990; Nadeau and Ferrari, 1998), that allow to relate the components of strain, stress and elasticity tensors.

First, for a full anisotropic solid, the following matrix representation can be constructed:

$$\begin{pmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{33}\\ \sigma_{23}\\ \sigma_{31}\\ \sigma_{12}\\ \sigma_{32}\\ \sigma_{13}\\ \sigma_{21}\end{pmatrix} = \begin{pmatrix} S_{1111} & S_{1122} & S_{1133} & S_{1123} & S_{1131} & S_{1112} & S_{1132} & S_{1113} & S_{1121}\\ S_{2211} & S_{2222} & S_{2233} & S_{2223} & S_{2231} & S_{2212} & S_{2232} & S_{2213} & S_{2221}\\ S_{3311} & S_{3322} & S_{3333} & S_{3323} & S_{3331} & S_{3312} & S_{3332} & S_{3313} & S_{3321}\\ S_{2311} & S_{2322} & S_{2333} & S_{2323} & S_{2331} & S_{2312} & S_{2332} & S_{2313} & S_{2321}\\ S_{3111} & S_{3122} & S_{3133} & S_{3123} & S_{3131} & S_{3112} & S_{3132} & S_{3113} & S_{3121}\\ S_{1211} & S_{1222} & S_{1233} & S_{1223} & S_{1231} & S_{1212} & S_{1232} & S_{1213} & S_{1221}\\ S_{3211} & S_{3222} & S_{3233} & S_{3223} & S_{3231} & S_{3212} & S_{3232} & S_{3213} & S_{3221}\\ S_{1311} & S_{1322} & S_{1333} & S_{1323} & S_{1331} & S_{1312} & S_{1332} & S_{1313} & S_{1321}\\ S_{2111} & S_{2122} & S_{2133} & S_{2123} & S_{2131} & S_{2112} & S_{2132} & S_{2113} & S_{2121}\end{pmatrix}\begin{pmatrix}\epsilon_{11}\\ \epsilon_{22}\\ \epsilon_{33}\\ \epsilon_{23}\\ \epsilon_{31}\\ \epsilon_{12}\\ \epsilon_{32}\\ \epsilon_{13}\\ \epsilon_{21}\end{pmatrix}. \tag{9}$$

Nevertheless, notation (9) does not make use of the minor and major symmetries of tensor $\mathbb{S}$. If these symmetries are considered, the extended formulation (9) can be abandoned and more convenient representations of Hooke's law are possible, which reduce the size of matrices and vectors. Among these, two are more widely adopted in the literature. The first, known as Voigt's notation (Love, 1944), reads:

$$\begin{pmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{33}\\ \sigma_{23}\\ \sigma_{31}\\ \sigma_{12}\end{pmatrix} = \begin{pmatrix} S_{1111} & S_{1122} & S_{1133} & 2S_{1123} & 2S_{1131} & 2S_{1112}\\ S_{1122} & S_{2222} & S_{2233} & 2S_{2223} & 2S_{2231} & 2S_{2212}\\ S_{1133} & S_{2233} & S_{3333} & 2S_{3323} & 2S_{3331} & 2S_{3312}\\ 2S_{1123} & 2S_{2223} & 2S_{3323} & 4S_{2323} & 4S_{2331} & 4S_{2312}\\ 2S_{1131} & 2S_{2231} & 2S_{3331} & 4S_{2331} & 4S_{3131} & 4S_{3112}\\ 2S_{1112} & 2S_{2212} & 2S_{3312} & 4S_{2312} & 4S_{3112} & 4S_{1212}\end{pmatrix}\begin{pmatrix}\epsilon_{11}\\ \epsilon_{22}\\ \epsilon_{33}\\ 2\epsilon_{23}\\ 2\epsilon_{31}\\ 2\epsilon_{12}\end{pmatrix}. \tag{10}$$

The presence of the multiplicative factor 2 applied to the shearing strains results from the exploitation of the minor symmetries. Moreover, the position of such a factor allows the stress and strain vectors to assume physical sense.

A second possible representation, poor of physical meaning but more attractive than the previous one by virtue of its tensorial properties, is the linear transformation in six dimensions (Walpole, 1984; Rychlewski, 1984):

$$\begin{pmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{33}\\ \sqrt{2}\sigma_{23}\\ \sqrt{2}\sigma_{31}\\ \sqrt{2}\sigma_{12}\end{pmatrix} = \begin{pmatrix} S_{1111} & S_{1122} & S_{1133} & \sqrt{2}S_{1123} & \sqrt{2}S_{1131} & \sqrt{2}S_{1112}\\ S_{1122} & S_{2222} & S_{2233} & \sqrt{2}S_{2223} & \sqrt{2}S_{2231} & \sqrt{2}S_{2212}\\ S_{1133} & S_{2233} & S_{3333} & \sqrt{2}S_{3323} & \sqrt{2}S_{3331} & \sqrt{2}S_{3312}\\ \sqrt{2}S_{1123} & \sqrt{2}S_{2223} & \sqrt{2}S_{3323} & 2S_{2323} & 2S_{2331} & 2S_{2312}\\ \sqrt{2}S_{1131} & \sqrt{2}S_{2231} & \sqrt{2}S_{3331} & 2S_{2331} & 2S_{3131} & 2S_{3112}\\ \sqrt{2}S_{1112} & \sqrt{2}S_{2212} & \sqrt{2}S_{3312} & 2S_{2312} & 2S_{3112} & 2S_{1212}\end{pmatrix}\begin{pmatrix}\epsilon_{11}\\ \epsilon_{22}\\ \epsilon_{33}\\ \sqrt{2}\epsilon_{23}\\ \sqrt{2}\epsilon_{31}\\ \sqrt{2}\epsilon_{12}\end{pmatrix}. \tag{11}$$

Note that in last notation the stress and strain tensors have been mapped into the six-dimensional space in the same manner. The difference between definitions (10) and (11) may appear shallow, but it is not indeed. It has been proved (Mehrabadi and Cowin, 1990) that the 6×6 matrix in (11) contains the components of a second-order tensor in the six-dimensional space, whereas it is well known (Nye, 1957; Hearmon, 1961; Fedorov, 1968) that the 6×6 matrix in (10) does not contain the components of a second-order tensor, and must be exclusively understood as a matrix.

However, both in classical and in the majority of modern literature on anisotropic solids, the choice (10) results to be the most widespread, despite its lack of tensorial character, and therefore it will be also adopted here. Anyway, this choice does not affect the results which follow.

In view of defining the material symmetries that will be considered in this paper, a special reference system is introduced. A 'principal' reference system for a material symmetry is defined as a system of coordinates in which the elasticity tensor shows the fewest number of independent non-zero components. An exception is the case of full anisotropy, where this definition is meaningless.

The material symmetries considered here are two of those symmetries such that, when the solid is subjected to a unit dipole acting along a material principal direction, no shear strains arise. The material symmetries fulfilling this requirement are those corresponding to the groups (Coleman and Noll, 1964; Gurtin, 1972; Huo and Del Piero, 1991): $\mathscr{C}_6 \equiv \mathscr{C}_7$ (cubic symmetry, characterized by three elastic constants), $\mathscr{C}_{10} \equiv \mathscr{C}_{11}$ (hexagonal symmetry, five elastic constants: *transverse isotropy*), $\mathscr{C}_5$ (tetragonal, six elastic constants) and $\mathscr{C}_3$ (orthorhombic, nine elastic constants: *orthotropy*).

In Sections 3 and 4, the behavior of Young's modulus for cubic and hexagonal material symmetries will be examined in detail, by following an order where an increasing number of independent elastic material constants is considered.

3. Cubic symmetry

The cubic case represents the symmetry with the lesser number, 3, of independent elastic constants among the crystallographic classes, excluding, of course, the case of isotropy. It corresponds to the symmetry group $\mathscr{C}_6$ coinciding, in the case of a symmetric $\mathbb{S}$ (hyperelasticity), with the symmetry group $\mathscr{C}_7$ (Huo and Del Piero, 1991). For the symmetry under investigation, the matrix representation of the elasticity tensor in (9), written in the principal reference system, and taking into account also the symmetries on $\mathbb{S}$ has this simpler form:

$$\begin{pmatrix} S_{1111} & S_{1122} & S_{1122} & 0 & 0 & 0 & 0 & 0 & 0 \\ S_{1122} & S_{1111} & S_{1122} & 0 & 0 & 0 & 0 & 0 & 0 \\ S_{1122} & S_{1122} & S_{1111} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & S_{2323} & 0 & 0 & S_{2323} & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{2323} & 0 & 0 & S_{2323} & 0 \\ 0 & 0 & 0 & 0 & 0 & S_{2323} & 0 & 0 & S_{2323} \\ 0 & 0 & 0 & S_{2323} & 0 & 0 & S_{2323} & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{2323} & 0 & 0 & S_{2323} & 0 \\ 0 & 0 & 0 & 0 & 0 & S_{2323} & 0 & 0 & S_{2323} \end{pmatrix}. \tag{12}$$

As explained in Section 2, here and in the following it is preferable to express the components of the elastic compliance tensor $\mathbb{S}$ in the Voigt's contracted notation which allows to construct, in the principal material Cartesian reference frame, the elastic matrix in the form:

$$\begin{pmatrix} s_{11} & s_{12} & s_{12} & 0 & 0 & 0 \\ s_{12} & s_{11} & s_{12} & 0 & 0 & 0 \\ s_{12} & s_{12} & s_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & s_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & s_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & s_{44} \end{pmatrix}, \tag{13}$$

where the reduced elastic coefficients are defined as $s_{11} = S_{1111} = S_{2222} = S_{3333}$, $s_{12} = S_{1122} = S_{1133} = S_{2233}$, $s_{44} = 4S_{2323} = 4S_{3131} = 4S_{1212}$ (see (10) and Nye, 1957; Hearmon, 1961; Sirotin and Chaskolkaïa, 1984).

Having assumed the existence of a positive definite elastic energy, some restrictions must be satisfied by the components of the reduced matrix (13). Therefore, application of Jordan's lemma to (13) leads to the following inequalities:

$$s_{11} > 0, \tag{14}$$

$$-\frac{s_{11}}{2} < s_{12} < s_{11}, \tag{15}$$

$$s_{44} > 0. \tag{16}$$

In terms of contracted notation, expression (4) for $1/E(\boldsymbol{n})$ gives:

$$\frac{1}{E(\boldsymbol{n})} = s_{12} + \frac{1}{2}s_{44} + \frac{1}{2}(2s_{11} - 2s_{12} - s_{44})\left(n_1^4 + n_2^4 + n_3^4\right) \tag{17}$$

which, making use of the identity:

$$n_1^4 + n_2^4 + n_3^4 = 1 - 2\left(n_1^2 n_2^2 + n_1^2 n_3^2 + n_2^2 n_3^2\right) \tag{18}$$

can be rewritten as:

$$\frac{1}{E(\boldsymbol{n})} = s_{11} - (2s_{11} - 2s_{12} - s_{44})\left(n_1^2 n_2^2 + n_1^2 n_3^2 + n_2^2 n_3^2\right). \tag{19}$$

As Eq. (19) shows, the Young's modulus for the cubic case depends on all three elastic constants s_{11}, s_{12} and s_{44}. For convenience, relation (19) is rewritten in the following form:

$$\frac{1}{E(\boldsymbol{n})} = s_{11} - \beta_1\left(n_1^2 n_2^2 + n_1^2 n_3^2 + n_2^2 n_3^2\right), \tag{20}$$

where a new coefficient, depending on material properties only,

$$\beta_1 := 2s_{11} - 2s_{12} - s_{44} \tag{21}$$

has been defined. Since the eigenvalues of (13) are, as one can easily check, $\lambda_1 = \lambda_2 = s_{11} - s_{12}$, $\lambda_3 = s_{11} + 2s_{12}$ and $\lambda_4 = \lambda_5 = \lambda_6 = s_{44}$, then the material parameter β_1 can also be written in this way

$$\beta_1 = 2\lambda_1 - \lambda_4 \tag{22}$$

and thus represents an invariant quantity associated with the material. It should be noticed that β_1 is related to the dimensionless Zener anisotropy factor, frequently used in the literature about cubic materials (Zener, 1955; Edington, 1974; Kelly et al., 2000):

$$B_{\mathrm{cub}} := \frac{2(s_{11} - s_{12})}{s_{44}} \tag{23}$$

by the following relationship:

$$\beta_1 = s_{44}(B_{\mathrm{cub}} - 1). \tag{24}$$

It should be also noticed that for $\beta_1 = 0$ (i.e., $B_{\mathrm{cub}} = 1$) the isotropic case is recovered. Moreover, the coefficient β_1 is not sign restricted, but, by virtue of inequalities (14)–(16), it is, in any case, subjected to the restrictions:

$$-s_{44} < \beta_1 < 3s_{11}. \tag{25}$$

Eqs. (24) and (25) provide the corresponding bounds on the dimensionless anisotropy factor B_{cub}:

$$0 < B_{\mathrm{cub}} < 3\frac{s_{11}}{s_{44}} + 1. \tag{26}$$

Definitions of alternate parameters to B_{cub} or to β_1 are given in the literature (Nadeau and Ferrari, 2001). As pointed out there, however, all these parameters assume a range (either bounded or not) of values, where the isotropic case does correspond to neither the *minimum* nor to the *maximum*, but to a point internal to the range itself: see, for instance, bounds (25) and (26). Therefore, they suffer of a lack of uniqueness in defining the 'absolute' degree of anisotropy. To overcome this ambiguity, Nadeau and Ferrari (2001)

introduce a new family of anisotropy parameters, valid for all elastic symmetry classes, which vanish in correspondence to isotropy, and assume otherwise only positive values.

Nevertheless, being a classification of cubic materials the objective of this section, it becomes crucial to distinguish among different anisotropy-induced behaviors. As it will appear soon, 'bilateral' parameters, like B_{cub} and β_1, are both effective and precisely needed. Indeed, definition (21) allows for the following mechanical interpretation.

By virtue of inequalities (14)–(16) one has $(s_{11} - s_{12}) > 0$ and $s_{44} > 0$; therefore, for relatively low values of s_{44} it follows that $\beta_1 > 0$, while relatively high values of s_{44} give $\beta_1 < 0$. This means that the material parameter β_1 can be thought of as a measure of the relative shear stiffness of the material. In other words, the following classification holds (see also Pedersen, 1989, for similar definitions referred to orthotropic bodies in plane elasticity):

$\beta_1 > 0$: cubic material with high relative shear stiffness

$\beta_1 < 0$: cubic material with low relative shear stiffness.

For the cubic symmetry and in contracted notation, the Lagrangian function (6) takes the form:

$$\mathscr{L} = s_{11} - \beta_1\left(n_1^2 n_2^2 + n_1^2 n_3^2 + n_2^2 n_3^2\right) + \lambda\left(n_1^2 + n_2^2 + n_3^2 - 1\right) \tag{27}$$

and the relevant explicit stationarity conditions (8) read:

$$\begin{cases} \left[-\beta_1\left(n_2^2 + n_3^2\right) + \lambda\right] n_1 = 0 \\ \left[-\beta_1\left(n_1^2 + n_3^2\right) + \lambda\right] n_2 = 0 \\ \left[-\beta_1\left(n_1^2 + n_2^2\right) + \lambda\right] n_3 = 0 \\ n_1^2 + n_2^2 + n_3^2 = 1. \end{cases} \tag{28}$$

Eq. (28) provide the necessary conditions for (20) to be stationary. In order to find all the directions $\boldsymbol{n}$ corresponding to critical points of $1/E(\boldsymbol{n})$, let's define an orthonormal basis $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ in the three-dimensional space and denote by n_1, n_2, n_3 the components of the unit vector $\boldsymbol{n} = n_1\boldsymbol{e}_1 + n_2\boldsymbol{e}_2 + n_3\boldsymbol{e}_3$. Then, one can distinguish three different situations.

First, if $\boldsymbol{n}$ is directed along a coordinate axis (i.e., if $\boldsymbol{n} = \pm\boldsymbol{e}_1$, or $\boldsymbol{n} = \pm\boldsymbol{e}_2$ or $\boldsymbol{n} = \pm\boldsymbol{e}_3$) then the following obvious solutions are obtained, respectively:

$$(1)\begin{cases} n_1^2 = 1 \\ n_2^2 = n_3^2 = 0 \\ \lambda = 0 \\ \dfrac{1}{E} = s_{11} \end{cases} \quad (2)\begin{cases} n_2^2 = 1 \\ n_1^2 = n_3^2 = 0 \\ \lambda = 0 \\ \dfrac{1}{E} = s_{11} \end{cases} \quad (3)\begin{cases} n_3^2 = 1 \\ n_1^2 = n_2^2 = 0 \\ \lambda = 0 \\ \dfrac{1}{E} = s_{11} \end{cases} \tag{29}$$

and, in these directions, the Young's modulus attains the same value $E = 1/s_{11}$, in agreement with the structure of the elastic matrix (13).

Second, if the unit vector $\boldsymbol{n}$ has two non-vanishing components (i.e., if $\boldsymbol{n} = n_2\boldsymbol{e}_2 + n_3\boldsymbol{e}_3$, or $\boldsymbol{n} = n_1\boldsymbol{e}_1 + n_3\boldsymbol{e}_3$ or $\boldsymbol{n} = n_1\boldsymbol{e}_1 + n_2\boldsymbol{e}_2$) it belongs to one of the coordinate planes. The corresponding solutions are, respectively:

$$(4)\begin{cases} n_1^2 = 0 \\ n_2^2 = n_3^2 = \dfrac{1}{2} \\ \lambda = \dfrac{\beta_1}{2} \\ \dfrac{1}{E} = s_{11} - \dfrac{\beta_1}{4} \end{cases} \quad (5)\begin{cases} n_2^2 = 0 \\ n_1^2 = n_3^2 = \dfrac{1}{2} \\ \lambda = \dfrac{\beta_1}{2} \\ \dfrac{1}{E} = s_{11} - \dfrac{\beta_1}{4} \end{cases} \quad (6)\begin{cases} n_3^2 = 0 \\ n_1^2 = n_2^2 = \dfrac{1}{2} \\ \lambda = \dfrac{\beta_1}{2} \\ \dfrac{1}{E} = s_{11} - \dfrac{\beta_1}{4} \end{cases} \tag{30}$$

and the Young's modulus depends also on the off-diagonal element s_{12} of (13) and on the shear modulus s_{44}, through the parameter β_1.

It should be noticed that in the last three cases the values of n_i^2 ($i = 1, 2, 3$) are independent of the elastic constants, and therefore the admissibility conditions $0 \leqslant n_i^2 \leqslant 1$ are automatically guaranteed. On the other hand, the positivity condition on $1/E$ is a priori satisfied if the parameter β_1 is expressed through Eq. (21), taking into account inequalities (14)–(16). Conversely, the condition $1/E > 0$ implies the restriction $\beta_1 < 4s_{11}$, weaker than the upper bound (25).

Finally, if $\boldsymbol{n}$ is a unit vector generically oriented, with all non-vanishing components ($\boldsymbol{n} = n_1\boldsymbol{e}_1 + n_2\boldsymbol{e}_2 + n_3\boldsymbol{e}_3$), the solution of (28) reads:

$$(7)\begin{cases} n_1^2 = n_2^2 = n_3^2 = \dfrac{1}{3} \\ \lambda = \dfrac{2\beta_1}{3} \\ \dfrac{1}{E} = s_{11} - \dfrac{\beta_1}{3}. \end{cases} \tag{31}$$

Once again, the Young's modulus depends on the elastic parameter s_{11} and on the anisotropy factor β_1. Also in this situation, the values n_i^2 ($i = 1, 2, 3$) are independent of the elastic coefficients, and the admissibility conditions $0 \leqslant n_i^2 \leqslant 1$ are satisfied by the solution. The positivity condition $1/E > 0$ is guaranteed beforehand if the β_1 factor is given through Eq. (21); on the contrary, the request $1/E = s_{11} - \beta_1/3 > 0$ implies $\beta_1 < 3s_{11}$, which coincides with the upper inequality restraint (25).

In conclusion, assuming $i, j, k = 1, 2, 3$ with $i \neq j \neq k$, the following statements hold:

1. $n_i^2 = 1, n_j^2 = n_k^2 = 0$, i.e., if the unit vector $\boldsymbol{n}$ matches in turn each coordinate axis. In this case the solution corresponds to six stationary points, each according to $n_i = \pm 1$ ($i = 1, 2, 3$), and the value of Young's modulus is such that:

$$\frac{1}{E} = s_{11}; \tag{32}$$

2. $n_i^2 = n_j^2 = 1/2, n_k^2 = 0$, then the unit vector $\boldsymbol{n}$ is directed in turn along the bisectors of each coordinate plane. Therefore the 12 stationary points are those for which $n_i = \pm 1/\sqrt{2},\ n_j = \pm 1/\sqrt{2},\ n_k = 0$, $(i, j, k = 1, 2, 3, i \neq j \neq k)$, and the Young's modulus is given by:

$$\frac{1}{E} = s_{11} - \frac{\beta_1}{4} = \frac{2s_{11} + 2s_{12} + s_{44}}{4}; \tag{33}$$

3. $n_i^2 = n_j^2 = n_k^2 = 1/3$; the solutions correspond to the case of unit vector $\boldsymbol{n}$ that trisects each octant of the coordinate system. This case produces eight stationary points: $n_i = \pm 1/\sqrt{3}, n_j = \pm 1/\sqrt{3}, n_k = \pm 1/\sqrt{3}$ (i, $j, k = 1, 2, 3, i \neq j \neq k$) and the elastic modulus is such that:

$$\frac{1}{E} = s_{11} - \frac{\beta_1}{3} = \frac{s_{11} + 2s_{12} + s_{44}}{3}. \tag{34}$$

Directions for local extrema of Young's modulus in solids with cubic symmetry (under uniaxial tension) have been obtained in a different way, looking for stationary values of the stored elastic energy, by Ostrowska-Maciejewska and Rychlewski (2001).

From Eqs. (32)–(34), the solutions can be ordered as follows:

$$\text{if } \beta_1 > 0 \text{ then:} \quad s_{11} > s_{11} - \frac{\beta_1}{4} > s_{11} - \frac{\beta_1}{3}; \tag{35}$$

$$\text{if } \beta_1 < 0 \text{ then:} \quad s_{11} < s_{11} - \frac{\beta_1}{4} < s_{11} - \frac{\beta_1}{3}. \tag{36}$$

It is worth noting that in both cases the solutions with $1/E = s_{11} - \beta_1/4$ (corresponding to the directions of $\boldsymbol{n}$ that bisect the coordinate planes) are always relative minima or relative maxima, for β_1 greater or less than zero, respectively. Consequently, by making use of Eqs. (32) and (34), it is possible to assert that:

$$\beta_1 > 0: \quad \begin{cases} E_{\max} = \dfrac{3}{s_{11} + 2s_{12} + s_{44}} \\ E_{\min} = \dfrac{1}{s_{11}}, \end{cases} \tag{37}$$

$$\beta_1 < 0: \quad \begin{cases} E_{\max} = \dfrac{1}{s_{11}} \\ E_{\min} = \dfrac{3}{s_{11} + 2s_{12} + s_{44}} \end{cases} \tag{38}$$

where $E_{\min} = (1/E)^{-1}_{\max}$ and $E_{\max} = (1/E)^{-1}_{\min}$. Moreover, it can be easily verified that the following relation between maximum and minimum values of Young's modulus holds (see also Hayes and Shuvalov, 1988):

$$\left(\frac{1}{E}\right)_{\max} - \left(\frac{1}{E}\right)_{\min} = \frac{|\beta_1|}{3}.$$

The relationships between the elements of matrix representation of tensor $\mathbb{S}$ and those of matrix representation of tensor $\mathbb{C}$ in Voigt's contracted notation are:

$$s_{11} = \frac{c_{11} - c_{12}}{c_{11}^2 + c_{11}c_{12} - 2c_{12}^2}, \tag{39}$$

$$s_{12} = \frac{c_{12}}{c_{11}^2 + c_{11}c_{12} - 2c_{12}^2}, \tag{40}$$

$$s_{44} = \frac{1}{c_{44}}, \tag{41}$$

where the following definitions hold: $c_{11} = C_{1111} = C_{2222} = C_{3333}$, $c_{12} = C_{1122} = C_{1133} = C_{2233}$ and $c_{44} = C_{2323} = C_{3131} = C_{1212}$ (Nye, 1957; Hearmon, 1961; Sirotin and Chaskolkaïa, 1984). In such a way, Eqs. (37) and (38) can be rewritten, in terms of stiffness coefficients, as:

$$\beta_1 > 0: \quad \begin{cases} E_{\max} = \dfrac{3(c_{11} + 2c_{12})c_{44}}{c_{11} + 2c_{12} + c_{44}} \\ E_{\min} = \dfrac{(c_{11} - c_{12})(c_{11} + 2c_{12})}{c_{11} + c_{12}}, \end{cases} \tag{42}$$

$$\beta_1 < 0: \quad \begin{cases} E_{\max} = \dfrac{(c_{11} - c_{12})(c_{11} + 2c_{12})}{c_{11} + c_{12}} \\ E_{\min} = \dfrac{3(c_{11} + 2c_{12})c_{44}}{c_{11} + 2c_{12} + c_{44}}. \end{cases} \tag{43}$$

Therefore, relations (37) and (38) define two categories of materials in the frame of the cubic system. To the first category, corresponding to the case $\beta_1 > 0$ (that is, when $B_{\mathrm{cub}} > 1$) belong metallic materials

like Pb, Cu, Ag, Au, Pd, Ni, Ge, Al (ordered here for decreasing values of the anisotropy factor B_{cub}), alkaline metals (Li, Na, K, Rb), Si and C in its crystalline state (diamond) and several compounds and alloys (Landolt and Börnstein, 1992). Materials like W, Mo, V, Cr and Nb (ordered again for decreasing values of the anisotropy factor B_{cub}) belong, instead, to the second category, for which $\beta_1 < 0$ (i.e., $B_{\mathrm{cub}} < 1$).

Young's modulus can be effectively represented by means of a spherical polar diagram, that is, with a surface generated by a vector whose length is proportional to the value of Young's modulus in the direction pointed by the vector itself.

As an example, function $E(\boldsymbol{n})$ is depicted in Fig. 1, where four representative cases of the circumstance $\beta_1 < 0$ (i.e., $B_{\mathrm{cub}} < 1$) are shown, exhibiting increasing values of the anisotropy factor. Maxima of Young's modulus are directed along directions parallel to the edges of a cube. In particular, in Fig. 1(a), an extreme case is depicted, in order to show how cubic materials (often awkwardly named as quasi-isotropic) might, in

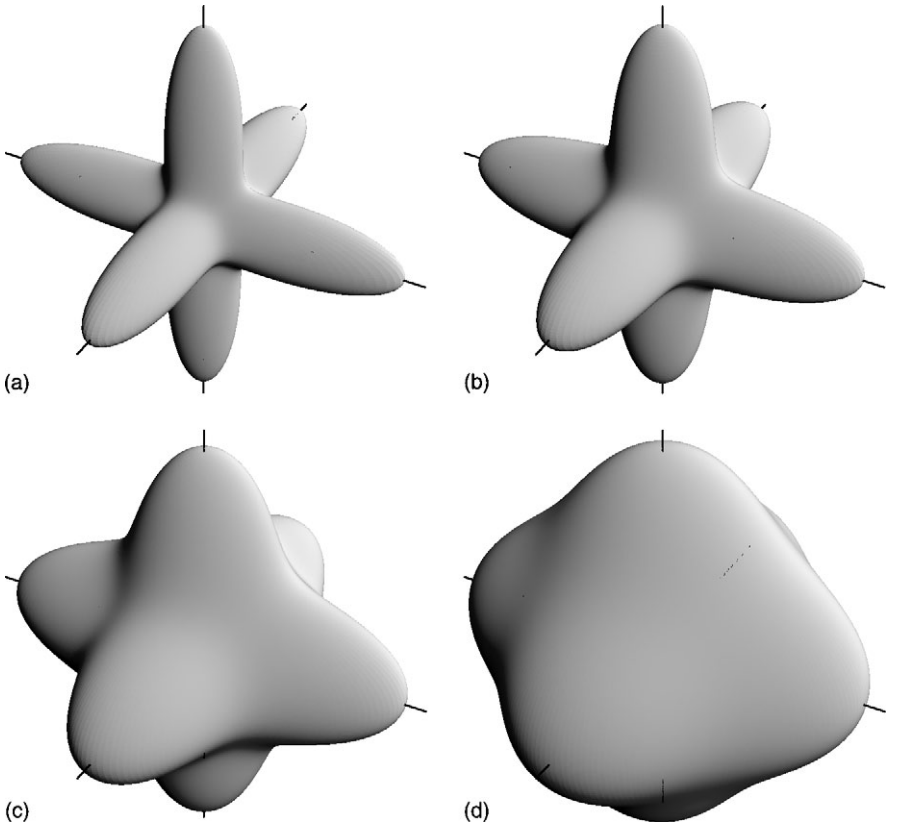

Fig. 1. Cubic system, characterized by parameter $\beta_1 < 0$ and dimensionless anisotropy factor $B_{\mathrm{cub}} < 1$: plots of $E(\boldsymbol{n})$. (a) GeTe–SnTe (mol% GeTe = 0): $\beta_1 = -84.54$ (TPa)$^{-1}$, $B_{\mathrm{cub}} = 0.18$. The compliance coefficients (units in (TPa)$^{-1}$) are: $s_{11} = 9.16$, $s_{44} = 103.10$, $s_{12} = -0.17$. Young's modulus: $E_{\min} = 0.027$ GPa and $E_{\max} = 0.109$ GPa. (b) RbBr (rubidium bromide): $\beta_1 = -186.02$ (TPa)$^{-1}$, $B_{\mathrm{cub}} = 0.29$. $s_{11} = 33.10$, $s_{44} = 262.00$, $s_{12} = -4.40$ (in (TPa)$^{-1}$). $E_{\min} = 0.010$, $E_{\max} = 0.030$ GPa. (c) Nb (niobium): $\beta_1 = -17.60$ (TPa)$^{-1}$, $B_{\mathrm{cub}} = 0.50$. $s_{11} = 6.56$, $s_{44} = 35.20$, $s_{12} = -2.29$ (in (TPa)$^{-1}$). $E_{\min} = 0.081$, $E_{\max} = 0.152$ GPa. (d) Cr–V (chromium–vanadium, Cr–0.67 at.% V): $\beta_1 = -2.98$ (TPa)$^{-1}$, $B_{\mathrm{cub}} = 0.70$. $s_{11} = 2.93$, $s_{44} = 9.93$, $s_{12} = -0.55$ (in (TPa)$^{-1}$). $E_{\min} = 0.255$, $E_{\max} = 0.341$ GPa. (Elastic coefficients taken from Landolt and Börnstein, 1992.)

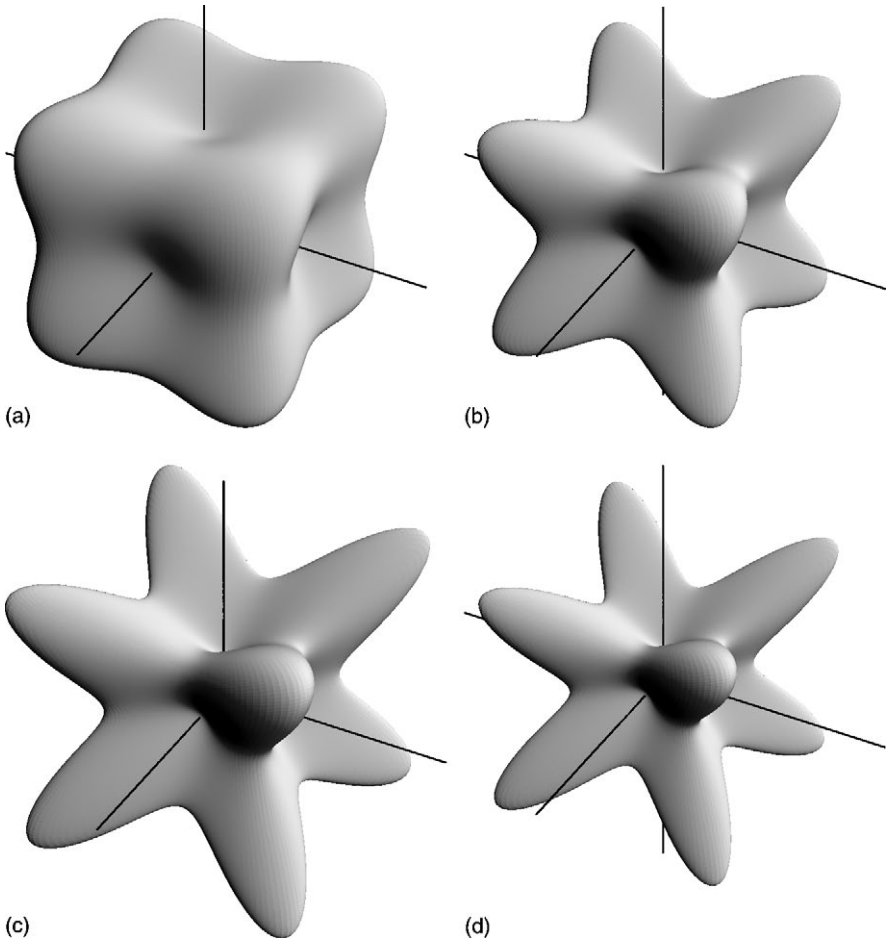

Fig. 2. Cubic system, characterized by parameter $\beta_1 > 0$ and dimensionless anisotropy factor $B_{\text{cub}} > 1$: plots of $E(\boldsymbol{n})$. (a) Cu–Au (copper–gold): $\beta_1 = 28.53$ $(\text{TPa})^{-1}$, $B_{\text{cub}} = 2.18$. The compliance coefficients (units in $(\text{TPa})^{-1}$) are: $s_{11} = 18.22$, $s_{44} = 24.09$, $s_{12} = -8.09$. Young's modulus: $E_{\min} = 0.055$ GPa and $E_{\max} = 0.115$ GPa. (b) Pb (lead): $\beta_1 = 205.40$ $(\text{TPa})^{-1}$, $B_{\text{cub}} = 4.02$. $s_{11} = 93.70$, $s_{44} = 68.00$, $s_{12} = 43.00$ (in $(\text{TPa})^{-1}$). $E_{\min} = 0.011$, $E_{\max} = 0.040$ GPa. (c) Rb (rubidium): $\beta_1 = 3235$ $(\text{TPa})^{-1}$, $B_{\text{cub}} = 6.18$. $s_{11} = 1331$, $s_{44} = 625$, $s_{12} = -600$ (in $(\text{TPa})^{-1}$). $E_{\min} = 0.0008$, $E_{\max} = 0.0040$ GPa. (d) Cs (cesium): $\beta_1 = 4200$ $(\text{TPa})^{-1}$, $B_{\text{cub}} = 7.21$. $s_{11} = 1676$, $s_{44} = 676$, $s_{12} = -762$ (in $(\text{TPa})^{-1}$). $E_{\min} = 0.0006$, $E_{\max} = 0.0036$ GPa. (Elastic coefficients taken from Landolt and Börnstein, 1992.)

some instances, exhibit a remarkably anisotropic behavior. There are, however, a few situations (one of which is that of Tungsten, W) where a cubic material can display a behavior very close to isotropy, namely when $\beta_1 \simeq 0$ or $B_{\text{cub}} \simeq 1$. The existence of such materials has also been theoretically predicted in literature (Rychlewski, 2000, 2001). In this case, the polar representation of Young's modulus approximates the shape of a sphere.

In Figs. 2–4, several cases for $\beta_1 > 0$ (or $B_{\text{hex}} > 1$) are shown, ordered again for increasing values of the anisotropy factor. The maximum values of Young's modulus happen to be directed, in this cases, along the diagonal of a cube. In particular, Fig. 4 highlights behaviors very far from isotropy, and Fig. 4(d) effectively points out an extreme real situation characterized by a strong anisotropy, close to the limit of material stability.

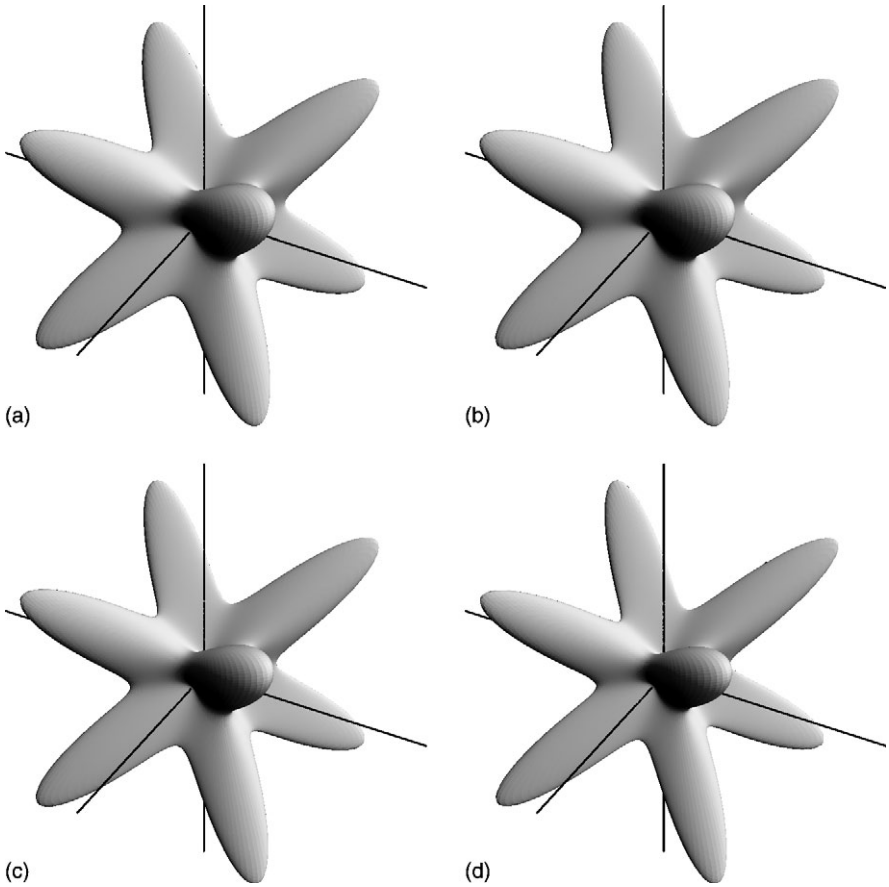

Fig. 3. Cubic system, characterized by parameter $\beta_1 > 0$ and dimensionless anisotropy factor $B_{\text{cub}} > 1$: plots of $E(\boldsymbol{n})$. (a) Li (lithium): $\beta_1 = 814\ (\text{TPa})^{-1}$, $B_{\text{cub}} = 8.83$. The compliance coefficients (units in $(\text{TPa})^{-1}$) are: $s_{11} = 315$, $s_{44} = 104$, $s_{12} = -144$. Young's modulus: $E_{\min} = 0.003$ GPa and $E_{\max} = 0.023$ GPa. (b) Al–Ni (aluminum–nickel) (at 63.2% Ni and at 273 K): $\beta_1 = 61.02\ (\text{TPa})^{-1}$, $B_{\text{cub}} = 9.05$. $s_{11} = 23.6$, $s_{44} = 7.58$, $s_{12} = -10.7$ (in $(\text{TPa})^{-1}$). $E_{\min} = 0.042$, $E_{\max} = 0.307$ GPa. (c) Cu–Al–Ni (copper–aluminum–nickel) (Cu–14 wt.% Al–4.1 wt.% Ni): $\beta_1 = 96.0\ (\text{TPa})^{-1}$, $B_{\text{cub}} = 10.23$. $s_{11} = 36.3$, $s_{44} = 10.4$, $s_{12} = -16.9$ (in $(\text{TPa})^{-1}$). $E_{\min} = 0.028$, $E_{\max} = 0.233$ GPa. (d) Cu–Al–Ni (copper–aluminum–nickel) (Cu–14.5 wt.% Al–3.15 wt.% Ni): $\beta_1 = 107.9\ (\text{TPa})^{-1}$, $B_{\text{cub}} = 12.12$. $s_{11} = 40.2$, $s_{44} = 9.7$, $s_{12} = -18.6$ (in $(\text{TPa})^{-1}$). $E_{\min} = 0.025$, $E_{\max} = 0.236$ GPa. (Elastic coefficients taken from Landolt and Börnstein, 1992.)

4. Hexagonal symmetry

The next case corresponds to the hexagonal symmetry class, group $\mathscr{C}_{10}$, which coincides with group $\mathscr{C}_{11}$ in the case of symmetric compliance tensor $\mathbb{S}$ (Huo and Del Piero, 1991). This case is representative of the *transversely isotropic* mechanical behavior. In this situation, the elasticity tensor is invariant under a reflection about a plane Π and for any rotation around an axis orthogonal to Π. Plane Π, and any plane parallel to it, are therefore planes of elastic isotropy. Without loss of generality, if the plane Π is assumed to coincide with the x_1–x_2 coordinate plane, and therefore the axis x_3 (along which the unit vector $\boldsymbol{e}_3$ lies) is assumed to be the axis of rotational symmetry, then the matrix structure (9) of tensor $\mathbb{S}$ simplifies to the form:

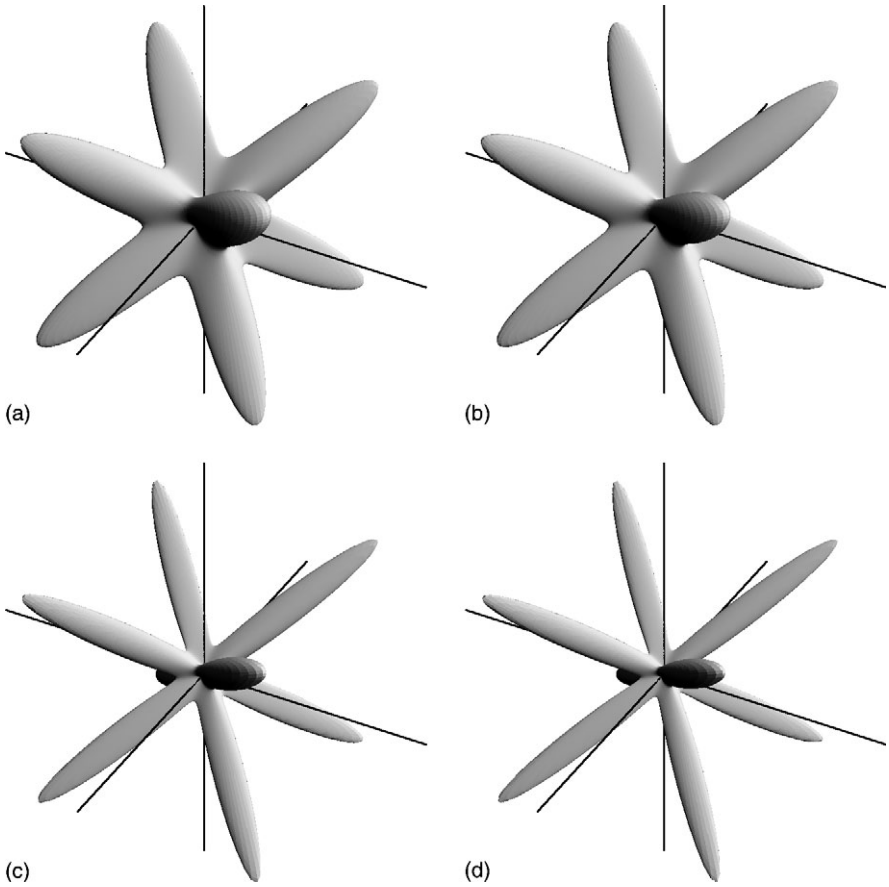

Fig. 4. Cubic system, characterized by parameter $\beta_1 > 0$ and dimensionless anisotropy factor $B_{\text{cub}} > 1$: plots of $E(\boldsymbol{n})$. (a) Cu–Al–Ni (copper–aluminum–nickel) (WQ(10): water quenching at 10 °C): $\beta_1 = 118.8\ (\text{TPa})^{-1}$, $B_{\text{cub}} = 13.12$. The compliance coefficients (units in $(\text{TPa})^{-1}$) are: $s_{11} = 43.7$, $s_{44} = 9.8$, $s_{12} = -20.6$. Young's modulus: $E_{\min} = 0.023$ GPa and $E_{\max} = 0.244$ GPa. (b) Al–Ni (aluminum–nickel) (at 60% Ni and at 273 K): $\beta_1 = 116.49\ (\text{TPa})^{-1}$, $B_{\text{cub}} = 15.02$. $s_{11} = 42.3$, $s_{44} = 8.31$, $s_{12} = -20.10$ (in $(\text{TPa})^{-1}$). $E_{\min} = 0.024$, $E_{\max} = 0.288$ GPa. (c) In–Tl (indium–thallium) (at 28.13% Tl): $\beta_1 = 3306\ (\text{TPa})^{-1}$, $B_{\text{cub}} = 28.55$. $s_{11} = 1145$, $s_{44} = 120$, $s_{12} = -568$ (in $(\text{TPa})^{-1}$). $E_{\min} = 0.0009$, $E_{\max} = 0.0233$ GPa. (d) In–Tl (indium–thallium) (at 25% Tl): $\beta_1 = 4222\ (\text{TPa})^{-1}$, $B_{\text{cub}} = 34.51$. $s_{11} = 1452$, $s_{44} = 126$, $s_{12} = -722$ (in $(\text{TPa})^{-1}$). $E_{\min} = 0.0007$, $E_{\max} = 0.0224$ GPa. (Elastic coefficients taken from Landolt and Börnstein, 1992.)

$$\begin{pmatrix} S_{1111} & S_{1122} & S_{1133} & 0 & 0 & 0 & 0 & 0 & 0 \\ S_{1122} & S_{1111} & S_{1133} & 0 & 0 & 0 & 0 & 0 & 0 \\ S_{1133} & S_{1133} & S_{3333} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & S_{2323} & 0 & 0 & S_{2323} & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{2323} & 0 & 0 & S_{2323} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{S_{1111}-S_{1122}}{2} & 0 & 0 & \frac{S_{1111}-S_{1122}}{2} \\ 0 & 0 & 0 & S_{2323} & 0 & 0 & S_{2323} & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{2323} & 0 & 0 & S_{2323} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{S_{1111}-S_{1122}}{2} & 0 & 0 & \frac{S_{1111}-S_{1122}}{2} \end{pmatrix}. \tag{44}$$

The matrix representation of the Voigt's reduced compliance coefficients (defined as $s_{11} = S_{1111} = S_{2222}$, $s_{12} = S_{1122}$, $s_{13} = S_{1133} = S_{2233}$, $s_{33} = S_{3333}$, $s_{44} = 4S_{2323} = 4S_{3131}$) looks as follows, when is expressed in the reference system of material symmetry:

$$\begin{pmatrix} s_{11} & s_{12} & s_{13} & 0 & 0 & 0 \\ s_{12} & s_{11} & s_{13} & 0 & 0 & 0 \\ s_{13} & s_{13} & s_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & s_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & s_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & 2(s_{11}-s_{12}) \end{pmatrix}. \tag{45}$$

The positive definiteness of the elastic energy establishes some restrictions on the reduced elastic coefficients in matrix (45), which are thus constrained by the following inequalities:

$$s_{11} > 0, \quad s_{33} > 0, \quad s_{44} > 0, \quad -s_{11} < s_{12} < s_{11}, \tag{46}$$

$$-\frac{1}{\sqrt{2}}\sqrt{s_{11}s_{33}}\sqrt{1+\frac{s_{12}}{s_{11}}} < s_{13} < \frac{1}{\sqrt{2}}\sqrt{s_{11}s_{33}}\sqrt{1+\frac{s_{12}}{s_{11}}}. \tag{47}$$

Young's modulus is expressed in terms of reduced elastic coefficients s_{ij} and of components of unit vector $\boldsymbol{n}$ through (4), which for transverse isotropy reads:

$$\frac{1}{E(\boldsymbol{n})} = s_{11} - \left[(s_{11}-s_{33})n_3^2 + (2s_{11}-2s_{13}-s_{44})\left(n_1^2+n_2^2\right)\right]n_3^2, \tag{48}$$

and shows the following features:

1. The elastic compliance coefficient s_{12}, which is relevant to the contraction in *transverse isotropy* plane, x_1–x_2, has no influence on the value of $1/E$.
2. As expected, the expression (48) of $1/E$ reveals an axis of rotational symmetry in the direction x_3, which is perpendicular to the transverse isotropy plane: indeed the components n_1 and n_2 of the unit vector $\boldsymbol{n}$ appear only under the form $n_1^2+n_2^2$. As a consequence, the representation of $1/E(\boldsymbol{n})$ in the three-dimensional space spanned by n_1, n_2, n_3 turns out to be a surface of revolution.

Eq. (48) can be rewritten as:

$$\frac{1}{E(\boldsymbol{n})} = s_{11} - \left[\alpha_2 n_3^2 + \beta_2\left(n_1^2+n_2^2\right)\right]n_3^2. \tag{49}$$

In expression (49), two material parameters, α_2 and β_2, have been introduced, and they are defined as:

$$\alpha_2 := s_{11} - s_{33}, \tag{50}$$

$$\beta_2 := 2s_{11} - 2s_{13} - s_{44}. \tag{51}$$

It should be also noticed that quantities α_2 and β_2 are not sign-restricted. Nevertheless, by virtue of inequalities $(46)_1$ and $(46)_2$, the following bounds must be satisfied:

$$-s_{33} < \alpha_2 < s_{11}. \tag{52}$$

No bounds can be prescribed for β_2, since limit values for s_{13} do not depend only on s_{11}, but also on s_{33} and on s_{12}, as it is clearly seen in inequalities (47), and these last two coefficients are independent of each other. However, if the condition $s_{13} < 0$ is met, as it happens for all materials listed in Landolt and Börnstein (1992) then the following lower bound holds for β_2:

$$-s_{44} < \beta_2.$$

By analogy with the cubic case, beside the material parameters α_2 and β_2 it is possible to define two dimensionless anisotropy factors:

$$A_{\text{hex}} = \frac{s_{11}}{s_{33}}, \tag{53}$$

$$B_{\text{hex}} = \frac{2(s_{11} - s_{13})}{s_{44}} \tag{54}$$

related to α_2 and β_2 as follows:

$$\alpha_2 = s_{33}(A_{\text{hex}} - 1), \tag{55}$$

$$\beta_2 = s_{44}(B_{\text{hex}} - 1). \tag{56}$$

Moreover, the bounds (52) imply the positivity of the dimensionless anisotropy factor A_{hex}; nothing can be assumed, in general, about B_{hex}, which is *not* sign restricted. However, when, as stated above, also the condition $s_{13} < 0$ is satisfied—and, indeed, this is the case for all materials belonging to hexagonal symmetry classes the authors are aware of—then it turns out that:

$$A_{\text{hex}} > 0 \quad \text{and} \quad B_{\text{hex}} > 0.$$

Under these *particular* circumstances it follows that factor A_{hex} can be interpreted as a measure of the anisotropy degree, while B_{hex}, as already done in the cubic case, as a measure of the relative shear stiffness. Indeed, by virtue of definition (51) and being $B_{\text{hex}} > 0$ and $s_{44} > 0$, one gets $(s_{11} - s_{13}) > 0$. Therefore, the material parameter β_2 allows to distinguish two classes of shear behavior:

$\beta_2 > 0$: transversely isotropic material with high relative shear stiffness

$\beta_2 < 0$: transversely isotropic material with low relative shear stiffness.

It should be noticed that, by virtue of definitions (50) and (51), in the isotropic case, i.e., when $s_{33} = s_{11}$ and $s_{44} = 2(s_{11} - s_{13})$, one has $\alpha_2 = \beta_2 = 0$; instead A_{hex} and B_{hex}—being genuinely anisotropy-related factors—reduce to unity in the case of isotropy. They might therefore be thought of as a generalization of the Zener anisotropy factor (23) used for cubic materials. There is however an important difference: while B_{cub} is sign-restricted, the same property is not inherited by B_{hex} because in general, as already pointed out, no bounds can be prescribed to β_2.

The above-mentioned dimensionless anisotropy factors can be usefully adopted to divide materials into different classes; however, in view of plotting the *loci* defined by Eq. (49) (or by its reciprocal), a more convenient choice of dimensionless parameters is:

$$A' := \frac{s_{33}}{s_{11}}, \quad B' := \frac{2s_{13} + s_{44}}{2s_{11}}. \tag{57}$$

Indeed, α_2 and β_2 are completely defined by the values of A', B' and s_{11} alone:

$$\alpha_2 = s_{11}(1 - A'), \quad \beta_2 = 2s_{11}(1 - B'), \tag{58}$$

while both s_{33} and s_{44} (i.e., one more parameter) must be specified when reconstructing α_2, β_2 from A_{hex} and B_{hex}, as shown in (55) and (56).

The Lagrangian function (6), for the transversely isotropic symmetry, can be written in the form:

$$\mathscr{L} = s_{11} - \left[\alpha_2 n_3^2 + \beta_2\left(n_1^2 + n_2^2\right)\right] n_3^2 + \lambda\left(n_1^2 + n_2^2 + n_3^2 - 1\right) \tag{59}$$

and the corresponding explicit stationarity conditions read:

$$\begin{cases} (-\beta_2 n_3^2 + \lambda) n_1 = 0 \\ (-\beta_2 n_3^2 + \lambda) n_2 = 0 \\ \left[-2\alpha_2 n_3^2 - \beta_2 (n_1^2 + n_2^2) + \lambda \right] n_3 = 0 \\ n_1^2 + n_2^2 + n_3^2 = 1. \end{cases} \tag{60}$$

It is now possible to carry out some assumption on the solutions, and to distinguish three different cases.

First, if $\boldsymbol{n}$ has only one non-vanishing component, that is if $\boldsymbol{n} = \pm \boldsymbol{e}_1$, or if $\boldsymbol{n} = \pm \boldsymbol{e}_2$, or even $\boldsymbol{n} = \pm \boldsymbol{e}_3$, the following solutions are obtained, respectively:

$$(1)\begin{cases} n_1^2 = 1 \\ n_2^2 = n_3^2 = 0 \\ \lambda = 0 \\ \dfrac{1}{E} = s_{11} \end{cases} \quad (2)\begin{cases} n_2^2 = 1 \\ n_1^2 = n_3^2 = 0 \\ \lambda = 0 \\ \dfrac{1}{E} = s_{11} \end{cases} \quad (3)\begin{cases} n_3^2 = 1 \\ n_1^2 = n_2^2 = 0 \\ \lambda = 2\alpha_2 \\ \dfrac{1}{E} = s_{11} - \alpha_2 = s_{33}. \end{cases} \tag{61}$$

It is easy to check that these solutions automatically satisfy both the admissibility conditions $0 \leqslant n_i^2 \leqslant 1$, $(i = 1, 2, 3)$ and the positivity condition $1/E > 0$.

Conversely, if $\boldsymbol{n}$ shows at the same time two non-vanishing components, that is if $\boldsymbol{n} = n_2 \boldsymbol{e}_2 + n_3 \boldsymbol{e}_3$, or $\boldsymbol{n} = n_1 \boldsymbol{e}_1 + n_3 \boldsymbol{e}_3$, or again $\boldsymbol{n} = n_1 \boldsymbol{e}_1 + n_2 \boldsymbol{e}_2$, one gets:

$$(4)\begin{cases} n_1^2 = 0 \\ n_2^2 = \dfrac{\beta_2 - 2\alpha_2}{2(\beta_2 - \alpha_2)} \\ n_3^2 = \dfrac{\beta_2}{2(\alpha_2 - \beta_2)} \\ \lambda = \dfrac{\beta_2^2}{2(\beta_2 - \alpha_2)} \\ \dfrac{1}{E} = s_{11} - \dfrac{\beta_2^2}{4(\beta_2 - \alpha_2)} \end{cases} \quad (5)\begin{cases} n_1^2 = \dfrac{\beta_2 - 2\alpha_2}{2(\beta_2 - \alpha_2)} \\ n_2^2 = 0 \\ n_3^2 = \dfrac{\beta_2}{2(\alpha_2 - \beta_2)} \\ \lambda = \dfrac{\beta_2^2}{2(\beta_2 - \alpha_2)} \\ \dfrac{1}{E} = s_{11} - \dfrac{\beta_2^2}{4(\beta_2 - \alpha_2)} \end{cases} \quad (6)\begin{cases} n_1^2 = \gamma^2 \\ n_2^2 = 1 - \gamma^2 \\ n_3^2 = 0 \\ \lambda = 0 \\ \dfrac{1}{E} = s_{11}. \end{cases} \tag{62}$$

Solutions (62) depend, through α_2 and β_2, on the elastic constants, so that the admissibility of such solutions is not guaranteed beforehand, but must be carefully checked, and this inspection is more complex than in the previously studied case of cubic symmetry. In other words, it is necessary to establish those conditions which must be satisfied by material parameters α_2 and β_2 in order to obtain solutions which are physically meaningful ($1/E > 0$ and $0 \leqslant n_i^2$, $i = 1, 2, 3$). To this purpose, in case (6), corresponding to the isotropy plane, it is straightforward to verify that the positivity of $1/E$ and the admissibility of n_i^2 (i.e., $n_1^2 > 0$, $n_2^2 > 0$) are met under the condition $0 < \gamma^2 < 1$.

The analogous proof is less trivial in cases (4) and (5). After some lengthy computations, here omitted, it can be shown that in cases (4) and (5) (in the latter by exchanging the roles played by n_1 and n_2) the admissibility conditions are those shown in Table 1, where the definition:

$$\beta_2^{\star} := 2 s_{11} \left(1 + \sqrt{1 - \frac{\alpha_2}{s_{11}}} \right) = 2 s_{11} \left(1 + \sqrt{A'} \right) \tag{63}$$

has been introduced.

Table 1
Hexagonal system: transversely isotropic solid

$\beta_2 \succ \alpha_2$	$\alpha_2 \succ 0$	$0 \leqslant n_2^2,\ n_3^2 \leqslant 1$	$\frac{1}{E} > 0$
$\beta_2 > \alpha_2$	$\alpha_2 > 0$	$\beta_2 \geqslant 2\alpha_2$	$2\alpha_2 \leqslant \beta_2 < \beta_2^\star$
	$\alpha_2 = 0$	$\beta_2 > 0$	$0 < \beta_2 < \beta_2^\star$
	$\alpha_2 < 0$	$\beta_2 \geqslant 0$	$0 \leqslant \beta_2 < \beta_2^\star$
$\beta_2 < \alpha_2$	$\alpha_2 > 0$	$\beta_2 \leqslant 0$	–
	$\alpha_2 = 0$	$\beta_2 < 0$	–
	$\alpha_2 < 0$	$\beta_2 \leqslant 2\alpha_2$	–

Admissibility conditions for solutions (4) and (5): Eqs. $(62)_1$ and $(62)_2$, for which $\boldsymbol{n} = n_2\boldsymbol{e}_2 + n_3\boldsymbol{e}_3$ and $\boldsymbol{n} = n_1\boldsymbol{e}_1 + n_3\boldsymbol{e}_3$, respectively.

Finally, in the most general case, that is when $\boldsymbol{n} = n_1\boldsymbol{e}_1 + n_2\boldsymbol{e}_2 + n_3\boldsymbol{e}_3$, the following solution is obtained:

$$(7)\begin{cases} n_1 = \gamma^2 \\ n_2^2 = \dfrac{\beta_2 - 2\alpha_2}{2(\beta_2 - \alpha_2)} - \gamma^2 \\ n_3^2 = \dfrac{\beta_2}{2(\beta_2 - \alpha_2)} \\ \lambda = \dfrac{\beta_2^2}{2(\beta_2 - \alpha_2)} \\ \dfrac{1}{E} = s_{11} - \dfrac{\beta_2^2}{4(\beta_2 - \alpha_2)}, \end{cases} \tag{64}$$

for any $\gamma^2 \in (0,1)$. Once again, by omitting the calculations, the admissibility conditions are obtained and shown in Table 2, where the following shorthand notation has been introduced:

$$\alpha_2^\star := \alpha_2 \frac{\gamma^2 - 1}{2\gamma^2 - 1} \tag{65}$$

together with definition (63).

By taking advantage of the afore-mentioned axis of rotational symmetry, when dealing with the surface (of revolution) defined by Eq. (48) or (49), analysis can be reduced to its intersection with a generic meridian plane. Indeed, on any plane which (i) is perpendicular to the plane Π of transverse isotropy and (ii) contains the axis of rotational symmetry, x_3, a point belonging to the (section of) surface (49) is completely defined by the spherical polar coordinates: radius, ρ, longitude (here: angle formed by the meridian plane with the positive x_1-axis), ϕ, and colatitude (here: angle formed by the radius with positive x_3-axis on the meridian plane), θ.

Table 2
Hexagonal system: transversely isotropic solid

$\beta_2 \succ \alpha_2$	$\alpha_2 \succ 0$	$n_1^2 = \gamma^2$	$0 \leqslant n_2^2,\ n_3^2 \leqslant 1$	$\frac{1}{E} > 0$
$\beta_2 > \alpha_2$	$\alpha_2 > 0$	$\gamma^2 < 1/2$	$\beta_2 \geqslant 2\alpha_2^\star$	$2\alpha_2^\star \leqslant \beta_2 < \beta_2^\star$
	$\alpha_2 = 0$	$\gamma^2 \leqslant 1/2$	$\beta_2 > 0$	$0 < \beta_2 < \beta_2^\star$
	$\alpha_2 < 0$	$\gamma^2 \leqslant 1/2$	$\beta_2 \geqslant 0$	$0 \leqslant \beta_2 < \beta_2^\star$
		$\gamma^2 > 1/2$	$0 \leqslant \beta_2 \leqslant 2\alpha_2^\star$	$0 \leqslant \beta_2 \leqslant 2\alpha_2^\star < \beta_2^\star$
$\beta_2 < \alpha_2$	$\alpha_2 > 0$	$\gamma^2 > 1/2$	$2\alpha_2^\star \leqslant \beta_2 \leqslant 0$	–
		$\gamma^2 \leqslant 1/2$	$\beta_2 \leqslant 0$	–
	$\alpha_2 = 0$	$\gamma^2 \leqslant 1/2$	$\beta_2 < 0$	–
	$\alpha_2 < 0$	$\gamma^2 < 1/2$	$\beta_2 \leqslant 2\alpha_2^\star$	–

Admissibility conditions for solution (7): Eqs. (64), for which $\boldsymbol{n} = n_1\boldsymbol{e}_1 + n_2\boldsymbol{e}_2 + n_3\boldsymbol{e}_3$.

It follows that:

$$\begin{aligned} n_1 &= \sin\theta\cos\phi \\ n_2 &= \sin\theta\sin\phi \\ n_3 &= \cos\theta, \end{aligned} \tag{66}$$

so that, by substituting Eqs. (66) into (49) the following expression of $1/E$ as a function of θ only is obtained:

$$\frac{1}{E(\boldsymbol{n})} = \frac{1}{E(\theta)} = s_{11} - (\alpha_2\cos^2\theta + \beta_2\sin^2\theta)\cos^2\theta. \tag{67}$$

Eq. (67) provides a complete description of the generic meridian section of the surface of revolution.

By virtue of mirror symmetry with respect to plane Π (i.e., plane x_1–x_2), analysis may be restricted to the range $0 \leqslant \theta \leqslant \pi/2$. It is then easy to realize that there are *always* two bounded extrema, the former corresponding to $\theta = \theta_1 = 0$ (pole), the latter to $\theta = \theta_2 = \pi/2$ (equator). To these points a stationary point corresponding to the value

$$\theta = \theta_\star = \cos^{-1}\sqrt{\frac{1}{2} + \frac{\alpha_2}{2(\beta_2 - \alpha_2)}} \tag{68}$$

must be added, but *only if* conditions listed in Table 1 or 2 are satisfied.

The corresponding values of $1/E$ turn out to be:

$$\begin{aligned} \frac{1}{E_1} &:= \frac{1}{E(\theta_1)} = s_{33} \\ \frac{1}{E_2} &:= \frac{1}{E(\theta_2)} = s_{11} \\ \frac{1}{E_\star} &:= \frac{1}{E(\theta_\star)} = s_{11} - \frac{\beta_2^2}{4(\beta_2 - \alpha_2)} = \frac{s_{11}s_{33} - (s_{13} + s_{44}/2)^2}{s_{11} + s_{33} - 2s_{13} - s_{44}}. \end{aligned} \tag{69}$$

In order to classify the stationary points of $1/E$, the following four cases will be separately studied.

(1) $\beta_2 > \alpha_2$ and $\alpha_2 \geqslant 0$, i.e., $B' < (1 + A')/2$ and $0 < A' \leqslant 1$

Since $\alpha_2 \geqslant 0$ it turns out, by (50), $1/E_1 \leqslant 1/E_2$, where the equal sign holds only when $\alpha_2 = 0$. In such conditions, the radius of the surface of revolution measured along the x_3 axis (i.e., the *polar* radius) equals the value measured on the Π plane, (i.e., the *equatorial* radius); for any other value of α_2 the polar radius is less than the equatorial radius, resulting in an oblate surface of revolution.

According to Eq. $(69)_3$ it is easy to acknowledge that point $\theta_\star$ corresponds to an absolute *minimum*; however, because of the restrictions listed in Table 1 (or, in Table 2, which provides the same results even though with some inessential intricacies) the solution is acceptable only within the range

$$2\alpha_2 \leqslant \beta_2 < \beta_2^\star, \tag{70}$$

where a strict inequality sign must be substituted in expression (70) when $\alpha_2 = 0$.

When within the range (70) the value of β_2 approaches the lower bound, the point $\theta_\star$ coincides with θ_1 and only two distinct stationary points (namely, θ_1 and θ_2) survive; if, instead, β_2 tends to the upper bound, the minimum value, which is attained when $\theta_\star \to (\pi/4)^-$ moves toward the origin, becoming smaller and smaller.

When $\alpha_2 = 0$ this circumstance happens for a value $\beta_2^\star = 4s_{11}$, corresponding, by virtue of inequalities (46) and (47), to fixing the values $s_{33} = s_{12} = s_{11}$; $s_{13} = -s_{11}$; $s_{44} = 0$, thus violating the positive definiteness of tensor $\mathbb{S}$.

It should be noticed that while β_2 increases within the range (70), the extremum point $\theta_\star$ moves from θ_1 in the direction of increasing colatitude angles (i.e., goes down from the pole along a meridian), but can never go beyond the value $\theta = \pi/4$, as clearly shown by Eq. (68).

The following conclusions might be therefore synthetically drawn:

(a) $\alpha_2 < \beta_2 < 2\alpha_2$ (i.e., $A' < B' < (1 + A')/2$): Function $1/E$ is monotonically increasing along the meridian from the pole, where it attains the minimum, s_{33}, to the equator, where it reaches the maximum, s_{11}. It should be emphasized that this circumstance can never happen if $\alpha_2 = 0$ (i.e., if $A' = 1$).
(b) $2\alpha_2 \leqslant \beta_2 < \beta_2^\star$ (i.e., $-\sqrt{A'} < B' \leqslant A'$): Function $1/E$ starts from a stationary value s_{33} (corresponding to the pole), then decreases along the meridian in order to attain its minimum value $1/E_\star$, corresponding to point $\theta_\star$, and finally increases, so that the maximum s_{11} is reached at equator. If $\alpha_2 = 0$ the values of s_{11} and s_{33} turn out to be equal (they are both maxima) and the absolute minimum defined by Eq. $(69)_3$ is always reached when $\theta_\star = \pi/4$.
(c) $\beta_2 \geqslant \beta_2^\star$ (i.e., $B' \leqslant -\sqrt{A'}$): This circumstance is not allowable, since it would require values of the elastic compliances which would violate the condition of a positive definite tensor $\mathbb{S}$.

(2) $\beta_2 > \alpha_2$ and $\alpha_2 < 0$, i.e., $B' < (1 + A')/2$ and $A' > 1$

Since, this time, $\alpha_2 < 0$ it turns out, by Eq. (50), that $1/E_1 > 1/E_2$.

The polar radius is now greater than the equatorial radius, resulting in a prolate surface of revolution.

According to Eq. $(69)_3$ it can be easily recognized that the stationary point $\theta_\star$ is still corresponding to an absolute minimum. Again the solution is acceptable only if the conditions listed in either Table 1 or Table 2 are fulfilled, i.e., within the range:

$$0 \leqslant \beta_2 < \beta_2^\star. \tag{71}$$

When β_2 reaches the lower bound in expression (71), point $\theta_\star$ coincides with θ_2, and only two distinct stationary points (namely, θ_1 and θ_2) remain; on the other hand, when the upper limit is approached by β_2, a vanishing minimum value, corresponding to $\theta_\star \to (\pi/4)^+$ is attained.

For $\alpha_2 \to 0$ this happens again for a value $\beta_2^\star = 4s_{11}$, which would imply, when inequalities (46) and (47) are considered, $s_{33} = s_{12} = s_{11}$; $s_{13} = -s_{11}$; $s_{44} = 0$, corresponding to a non-positive-definite elastic compliance tensor $\mathbb{S}$.

It should be noticed that, when β_2 is increasing within the range (71), the minimum point, $\theta_\star$ goes back up along the meridian from the equator (i.e., from $\theta = \pi/2$) to the pole, but cannot move beyond the value $\theta = \pi/4$.

These conditions need to be considered separately:

(a) $\alpha_2 < \beta_2 < 0$ (i.e., $1 < B' < (1 + A')/2$): Function $1/E$ is monotonically decreasing along the meridian from the maximum s_{33} (corresponding to the pole) to the minimum s_{11} (when it attains the equator); this condition, however, may occur only when $\alpha_2 < 0$.
(b) $0 \leqslant \beta_2 < \beta_2^\star$ (i.e., $-\sqrt{A'} < B' \leqslant 1$): Function $1/E$ starts from stationary value s_{11} (corresponding to the equator), and decreases—while going back up along the meridian—until it reaches in $\theta_\star$ its absolute minimum, $1/E_\star$. When further decreasing values of θ are considered, it starts increasing and attains its maximum, s_{33}, as soon as it reaches the pole. When $\alpha_2 \to 0^-$ one finds that s_{11} approaches the maximum value, s_{33}, while the absolute minimum occurs, as it is apparent from Eq. (68), for $\theta_\star \to (\pi/4)^+$.
(c) $\beta_2 \geqslant \beta_2^\star$ (i.e $B' \leqslant -\sqrt{A'}$): This circumstance is again not allowable, since it would require elastic compliance coefficients which would make tensor $\mathbb{S}$ non-positive definite.

(3) $\beta_2 < \alpha_2$ and $\alpha_2 \geqslant 0$, i.e., $B' > (1 + A')/2$ and $0 < A' \leqslant 1$

Since $\alpha_2 \geqslant 0$ it results, like in case 1 above, $1/E_1 \leqslant 1/E_2$, where the equal sign holds only when $\alpha_2 = 0$: again an oblate surface of revolution is obtained.

However, by looking at Eq. $(69)_3$ it turns out that an absolute maximum is now corresponding to the stationary point $\theta_\star$, provided that the following constraint, descending from Table 1 (or Table 2) is satisfied:

$$\beta_2 \leqslant 0, \tag{72}$$

where a strict inequality must be enforced if $\alpha_2 = 0$.

When the upper bound is approached by β_2 in expression (72), point $\theta_\star$ according to Eq. (68) coincides with θ_2, and only two distinct stationary points (namely, θ_1 and θ_2) are left. One should carefully notice, however, that—differently from cases 1 and 2 above—there are no more lower bounds on β_2: it can therefore decrease boundlessly without violating any restriction ensuring positive definiteness of tensor $\mathbb{S}$.

When β_2 decreases within the range (72) the stationary point $\theta_\star$ starts moving upwards along the meridian from the equator (i.e., from $\theta = \pi/2$) but is not allowed to go back up, as prescribed by Eq. (68), beyond the value $\theta = \pi/4$ (which is approached from above).

These brief conclusions can be drawn:

(a) $0 < \beta_2 < \alpha_2$ (i.e., $(1 + A')/2 < B' < 1$): Function $1/E$ is monotonically increasing with θ, i.e., moving downwards along the meridian from the pole (where the minimum s_{33} is attained) to the equator, where the maximum, s_{11} is reached; this circumstance is never possible if $\alpha_2 = 0$.
(b) $\beta_2 \leqslant 0$ (i.e., $B' \geqslant 1$): Function $1/E$ starts from stationary value s_{11} (corresponding to the equator) and, moving upwards along the meridian, is increasing until point $\theta_\star$, where it reaches its maximum $(69)_3$; after that it begins to decrease until it reaches the pole, where the minimum value, s_{33} is attained. When $\alpha_2 = 0$ it turns out that the values of s_{11} and s_{33} coincide (they are both minima), while the absolute maximum $1/E_\star$ is reached when $\theta_\star = \pi/4$. For any non-vanishing value of α_2 the maximum $\theta_\star \to (\pi/4)^+$ for decreasing values of β_2, as prescribed by Eq. (68).

(4) $\beta_2 < \alpha_2$ and $\alpha_2 < 0$ i.e., $B' > (1 + A')/2$ and $A' > 1$

It results, as in case 2 above, $\alpha_2 < 0$, so that $1/E_1 > 1/E_2$ and a prolate surface of revolution is obtained.

According to Eq. $(69)_3$ the stationary point $\theta_\star$ is again, as in case 3 above, an absolute maximum, provided that restriction deduced from Table 1 (or from Table 2) are enforced, i.e.:

$$\beta_2 \leqslant 2\alpha_2, \tag{73}$$

where it is always $\alpha_2 < 0$.

When β_2 approaches the upper bound of constraint (73) point $\theta_\star$ coincides with θ_2 so that only two distinct stationary points (θ_2 and θ_1) are left; however, no lower bounds need to be enforced on β_2, which appears therefore, as in case 3, to be unbounded from below. Moreover, as β_2 decreases, the point $\theta_\star$ where maximum is attained starting from the pole (i.e., $\theta = 0$) moves downwards along the meridian, according to Eq. (68) but cannot go beyond the value $\theta = \pi/4$, approaching this value from below.

Only these two circumstances can arise:

(a) $2\alpha_2 < \beta_2 < \alpha_2$ (i.e., $(1 + A')/2 < B' < A'$): Function $1/E$ is monotonically decreasing along the meridian ranging from maximum s_{33} (corresponding to the pole) to minimum s_{11} (corresponding to the equator).
(b) $\beta_2 \leqslant 2\alpha_2$: (i.e., $B' \geqslant A'$): Function $1/E$ starts from value s_{33} (corresponding to the pole), and for increasing values of θ (i.e., moving downwards along the meridian) increases until point $\theta_\star$, where it reaches its

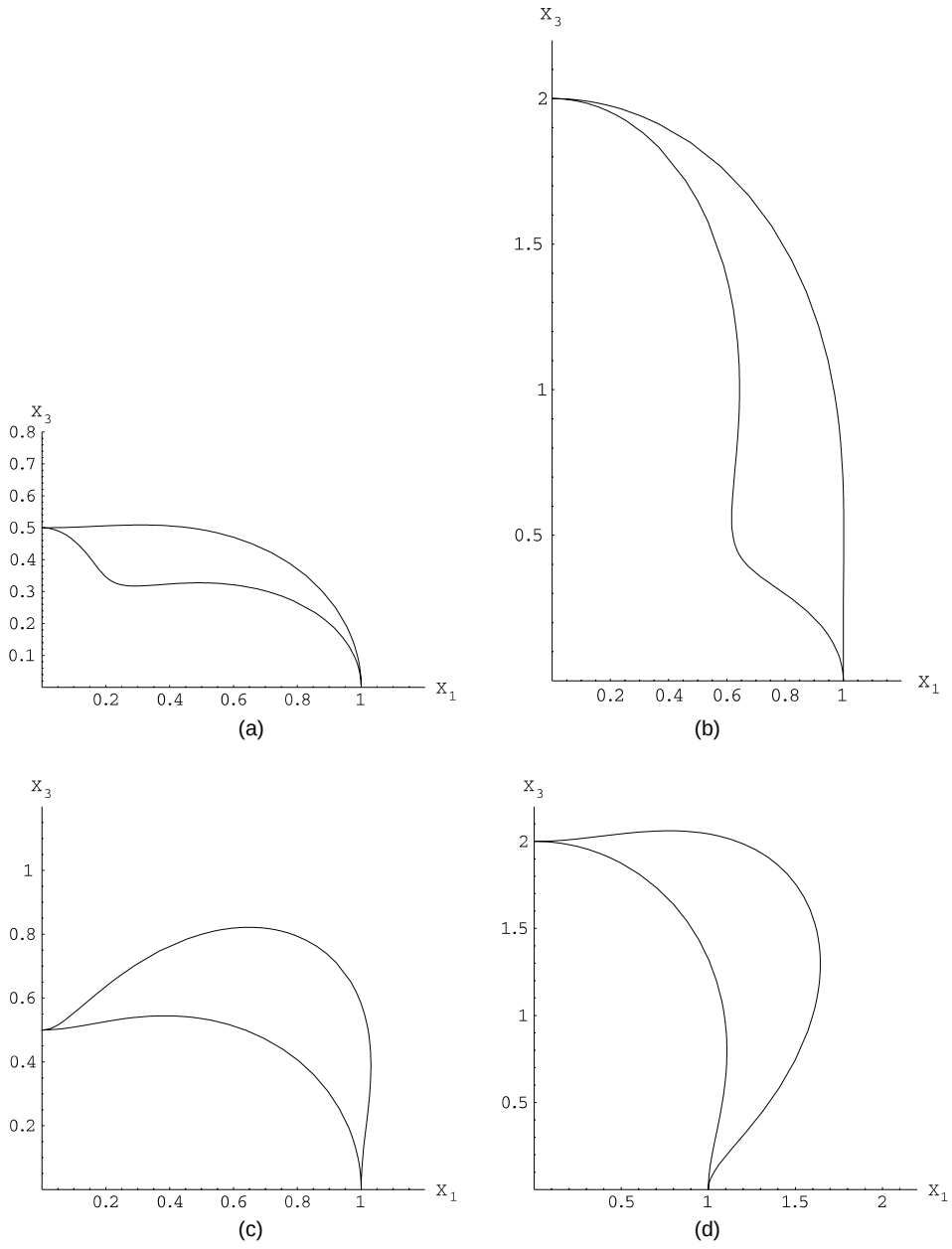

Fig. 5. Hexagonal system, meridian sections of function $1/E(\boldsymbol{n})$ expressed in dimensionless variables A' and B' for $0 \leqslant \theta \leqslant \pi/2$. (a) Case 1 ($\beta_2 > \alpha_2$ and $\alpha_2 \geqslant 0$) $A' = 0.50$; $B' = 0.65, 0.15$ (from outer to inner). (b) Case 2 ($\beta_2 > \alpha_2$ and $\alpha_2 < 0$) $A' = 2.00$; $B' = 1.25, 0.25$ (from outer to inner). (c) Case 3 ($\beta_2 < \alpha_2$ and $\alpha_2 \geqslant 0$) $A' = 0.50$; $B' = 1.50, 0.75$ (from outer to inner). (d) Case 4 ($\beta_2 < \alpha_2$ and $\alpha_2 < 0$) $A' = 2.00$; $B' = 3.00, 1.55$ (from outer to inner).

maximum value; afterwards it decreases reaching its minimum, s_{11}, corresponding to the equator. When $\alpha_2 \to 0^-$ the values of s_{33} and s_{11} tend to become equal (they are both minima), while the maximum,

$1/E_\star$, happens to correspond to point $\theta_\star = \pi/4$. For any non-vanishing value of α_2, according to Eq. (68), point $\theta_\star \to (\pi/4)^-$ as β_2 decreases.

The meridian section corresponding to the four different cases outlined above (and for both circumstances of having two or three stationary points in the range $0 \leqslant \theta \leqslant \pi/2$) are depicted in Fig. 5.

When switching from the inverse of Young's modulus, $1/E$, to Young's modulus, E, the same four cases presented above occur: the role of maxima and minima are exchanged, while angles preserve their value. It is indeed:

$$E_{\max} = \frac{1}{(1/E)_{\min}}; \quad E_{\min} = \frac{1}{(1/E)_{\max}}. \tag{74}$$

Taking into account Eqs. (69) and (74), the following shorthand notations can be introduced:

$$\begin{aligned} E_1 &:= E(\theta_1) = \frac{1}{s_{33}} \\ E_2 &:= E(\theta_2) = \frac{1}{s_{11}} \\ E_\star &:= E(\theta_\star) = \frac{s_{11} + s_{33} - 2s_{13} - s_{44}}{s_{11}s_{33} - (s_{13} + s_{44}/2)^2}. \end{aligned} \tag{75}$$

The elements of matrix representation of tensor $\mathbb{S}$ and those of matrix representation of tensor $\mathbb{C}$ are however linked; in the Voigt's contracted notation the relevant relations for the hexagonal symmetry are:

$$\begin{aligned} s_{11} &= \frac{-c_{13}^2 + c_{11}c_{33}}{(c_{11} - c_{12})(-2c_{13}^2 + (c_{11} + c_{12})c_{33})} \\ s_{12} &= \frac{c_{13}^2 - c_{12}c_{33}}{(c_{11} - c_{12})(-2c_{13}^2 + (c_{11} + c_{12})c_{33})} \\ s_{13} &= \frac{c_{13}}{2c_{13}^2 - (c_{11} + c_{12})c_{33}} \\ s_{33} &= \frac{c_{11} + c_{12}}{-2c_{13}^2 + (c_{11} + c_{12})c_{33}} \\ s_{44} &= \frac{1}{c_{44}}, \end{aligned} \tag{76}$$

where the following definitions (Nye, 1957; Hearmon, 1961; Sirotin and Chaskolkaïa, 1984) allow to express them as tensorial components: $c_{11} = C_{1111} = C_{2222}$, $c_{33} = C_{3333}$, $c_{12} = C_{1122}$, $c_{13} = C_{1133} = C_{2233}$ and $c_{44} = C_{2323} = C_{3131}$.

Alternatively, if expressions (76) are used, Eqs. (75) can be rewritten, in terms of stiffness coefficients, as follows:

$$\begin{aligned} E_1 &= \frac{-2c_{13}^2 + (c_{11} + c_{12})c_{33}}{c_{11} + c_{12}} \\ E_2 &= \frac{(c_{11} - c_{12})\left(-2c_{13}^2 + (c_{11} + c_{12})c_{33}\right)}{-c_{13}^2 + c_{11}c_{33}} \\ E_\star &= \frac{4c_{44}\left((c_{12}^2 - c_{11}^2)(c_{33} - c_{44}) - c_{13}^2c_{44} - 2c_{12}c_{13}(c_{13} + c_{44}) + c_{11}\left(2c_{13}^2 + 2c_{13}c_{44} + c_{33}c_{44}\right)\right)}{-c_{11}^2c_{33} + c_{12}(-2c_{13}^2 + c_{12}c_{33} - 4c_{13}c_{44}) + 2c_{11}(c_{13}^2 + 2c_{13}c_{44} + 2c_{44}^2)}. \end{aligned} \tag{77}$$

It should be noticed that all five elastic stiffness coefficients c_{11}, c_{12}, c_{13}, c_{33}, c_{44} enter the expressions (77), differently from compliance coefficients: only four of them, indeed, define the material response in terms of Young's modulus which, as already stated, is not influenced by the coefficient s_{12} corresponding to the contraction in the plane of transverse isotropy.

By virtue of Eqs. (75)—or, ad libitum (77)—the four different classes of mechanical behavior for hexagonal symmetry can be characterized as follows:

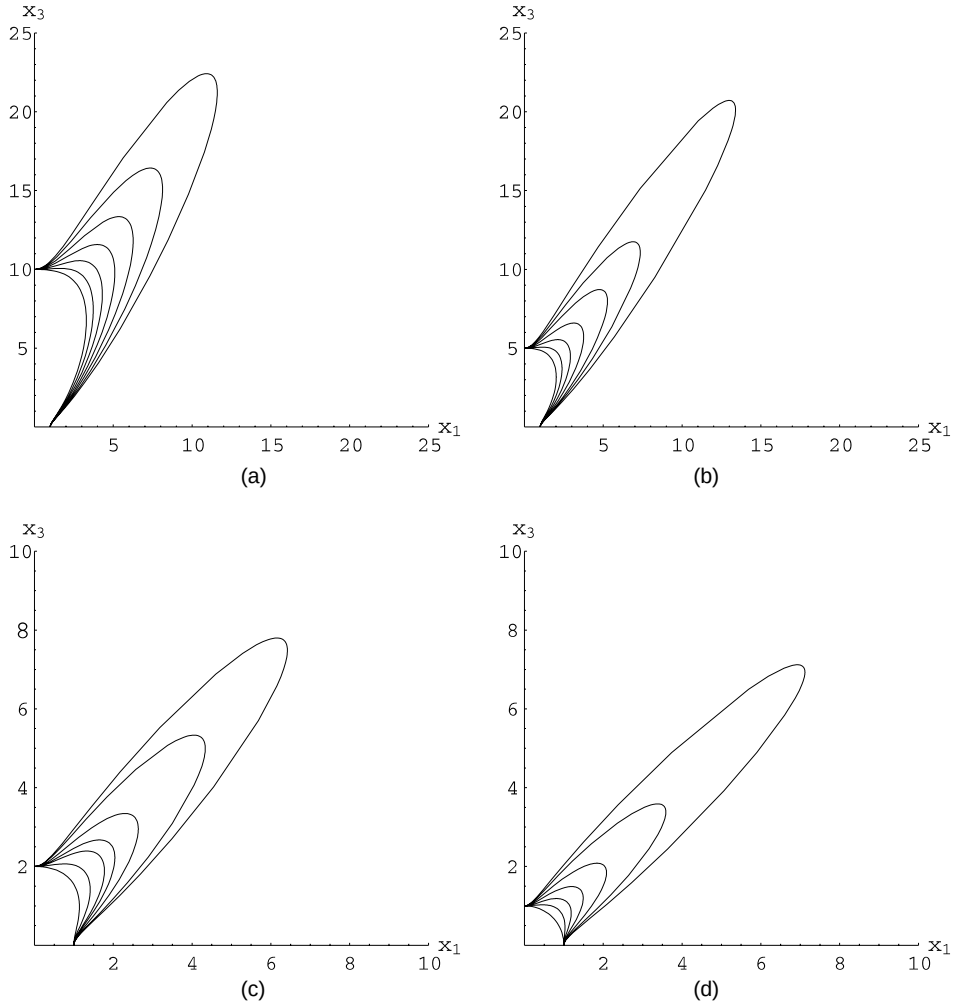

Fig. 6. Hexagonal system, case 1, characterized by $\beta_2 > \alpha_2$ and $\alpha_2 \geqslant 0$. Meridian sections of function $E(\boldsymbol{n})$ expressed in dimensionless variables A' and B' for $0 \leqslant \theta \leqslant \pi/2$. (a) $A' = 0.10$; $B' = -0.20, -0.15, -0.10, -0.05, 0.00, 0.05, 0.10$ (from outer to inner). (b) $A' = 0.20$; $B' = -0.35, -0.27, -0.20, -0.10, 0.00, 0.10, 0.20$ (from outer to inner). (c) $A' = 0.50$; $B' = -0.50, -0.40, -0.20, -0.05, 0.05, 0.25, 0.50$ (from outer to inner). (d) $A' = 1.00$; $B' = -0.80, -0.60, -0.30, 0.00, 0.30, 0.60, 0.99$ (from outer to inner).

1. $\beta_2 > \alpha_2$ and $\alpha_2 \geqslant 0$, i.e., $B' < (1 + A')/2$ and $0 < A' \leqslant 1$.

β_2	B'	$E_{\max}$	$\theta_{\max}$	E_{med}	θ_{med}	$E_{\min}$	$\theta_{\min}$
$\alpha_2 < \beta_2 < 2\alpha_2$	$A' < B' < (1 + A')/2$	E_1	θ_1	–	–	E_2	θ_2
$2\alpha_2 \leqslant \beta_2 < \beta_2^{\star}$	$-\sqrt{A'} < B' \leqslant A'$	$E_{\star}$	$\theta_{\star}$	E_1	θ_1	E_2	θ_2

θ_{med} and E_{med} in table above represent the third stationary point and the corresponding value of Young's modulus, respectively.

Meaningful examples of meridian sections for this class (restricted to the circumstance where all three stationary points exist in the range $0 \leqslant \theta \leqslant \pi/2$) are depicted in Fig. 6.

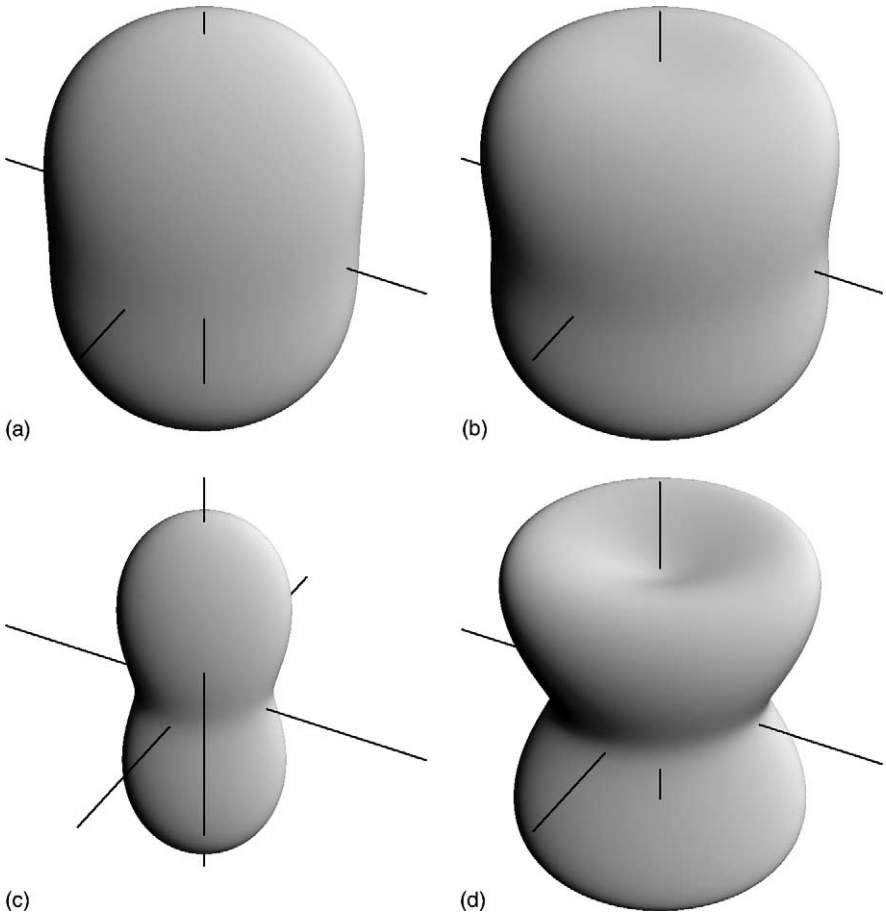

Fig. 7. Hexagonal system: plots of $E(\boldsymbol{n})$ for case 1, characterized by $\beta_2 > \alpha_2$ and $\alpha_2 \geqslant 0$. (a) WC (tungsten monocarbide). $\alpha_2 = 0.47$, $\beta_2 = 0.97$ (TPa)$^{-1}$, $A_{\text{hex}} = 1.388$, $B_{\text{hex}} = 1.318$. The compliance coefficients (units in (TPa)$^{-1}$) are: $s_{11} = 1.68$, $s_{33} = 1.21$, $s_{44} = 3.05$, $s_{12} = -0.47$, $s_{13} = -0.33$. Young's modulus: $E_{\min} = 0.595$ GPa and $E_{\max} = 0.827$ GPa. (b) Tc (technetium): $\alpha_2 = 0.30$ and $\beta_2 = 2.50$ (TPa)$^{-1}$. $A_{\text{hex}} = 1.103$ and $B_{\text{hex}} = 1.439$. $s_{11} = 3.2$, $s_{33} = 2.9$, $s_{44} = 5.7$, $s_{12} = -1.1$ and $s_{13} = -0.9$ (in (TPa)$^{-1}$). $E_{\min} = 0.313$ GPa and $E_{\max} = 0.402$ GPa. (c) MnAs (manganese arsenide): $\alpha_2 = 16.70$ and $\beta_2 = 27.00$ (TPa)$^{-1}$. $A_{\text{hex}} = 2.796$ and $B_{\text{hex}} = 1.931$. $s_{11} = 26.0$, $s_{33} = 9.3$, $s_{44} = 29.0$, $s_{12} = -5.0$ and $s_{13} = -2.0$ (in (TPa)$^{-1}$). $E_{\min} = 0.038$ GPa and $E_{\max} = 0.120$ GPa. (d) Co–Ni (cobalt–nickel): $\alpha_2 = 1.05$ and $\beta_2 = 7.03$ (TPa)$^{-1}$. $A_{\text{hex}} = 1.337$ and $B_{\text{hex}} = 1.520$. $s_{11} = 4.17$, $s_{33} = 3.12$, $s_{44} = 13.51$, $s_{12} = -18.8$ and $s_{13} = -6.1$ (in (TPa)$^{-1}$). $E_{\min} = 0.240$ GPa and $E_{\max} = 0.475$ GPa. (Elastic coefficients taken from Landolt and Börnstein, 1992.)

Be and $BaTiO_3$ among other materials, are representative of this class; the spherical polar diagram for some other real material belonging to this class are shown in Fig. 7.

2. $\beta_2 > \alpha_2$ and $\alpha_2 < 0$, i.e., $B' < (1 + A')/2$ and $A' > 1$.

β_2	B'	$E_{\max}$	$\theta_{\max}$	E_{med}	θ_{med}	$E_{\min}$	$\theta_{\min}$
$\alpha_2 < \beta_2 < 0$	$1 < B' < (1 + A')/2$	E_2	θ_2	–	–	E_1	θ_1
$0 \leqslant \beta_2 < \beta_2^{\star}$	$-\sqrt{A'} < B' \leqslant 1$	$E_{\star}$	$\theta_{\star}$	E_2	θ_2	E_1	θ_1

Meaningful examples of meridian sections for this class (restricted to the circumstance where all three stationary points exist in the range $0 \leqslant \theta \leqslant \pi/2$) are depicted in Fig. 8.

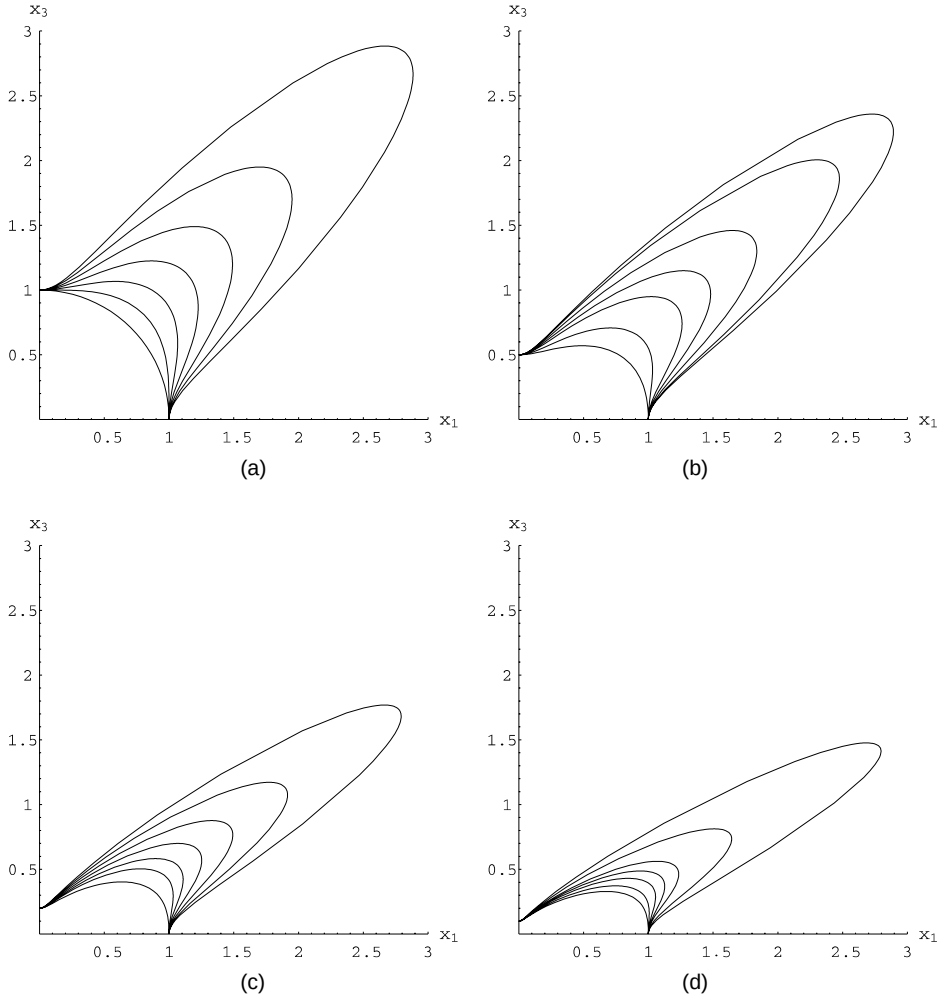

Fig. 8. Hexagonal system, case 2, characterized by $\beta_2 > \alpha_2$ and $\alpha_2 < 0$. Meridian sections of function $E(\boldsymbol{n})$ expressed in dimensionless variables A' and B' for $0 \leqslant \theta \leqslant \pi/2$. (a) $A' = 1.00$; $B' = -0.50, -0.25, 0.00, 0.25, 0.50, 0.75, 1.00$ (from outer to inner). (b) $A' = 2.00$; $B' = -0.85, -0.75, -0.50, -0.25, 0.00, 0.50, 1.00$ (from outer to inner). (c) $A' = 5.00$; $B' = -1.50, -1.25, -0.75, -0.375, 0.00, 0.35, 1.00$ (from outer to inner). (d) $A' = 10.00$; $B' = -2.25, -1.50, -0.75, -0.375, 0.00, 0.50, 1.00$ (from outer to inner).

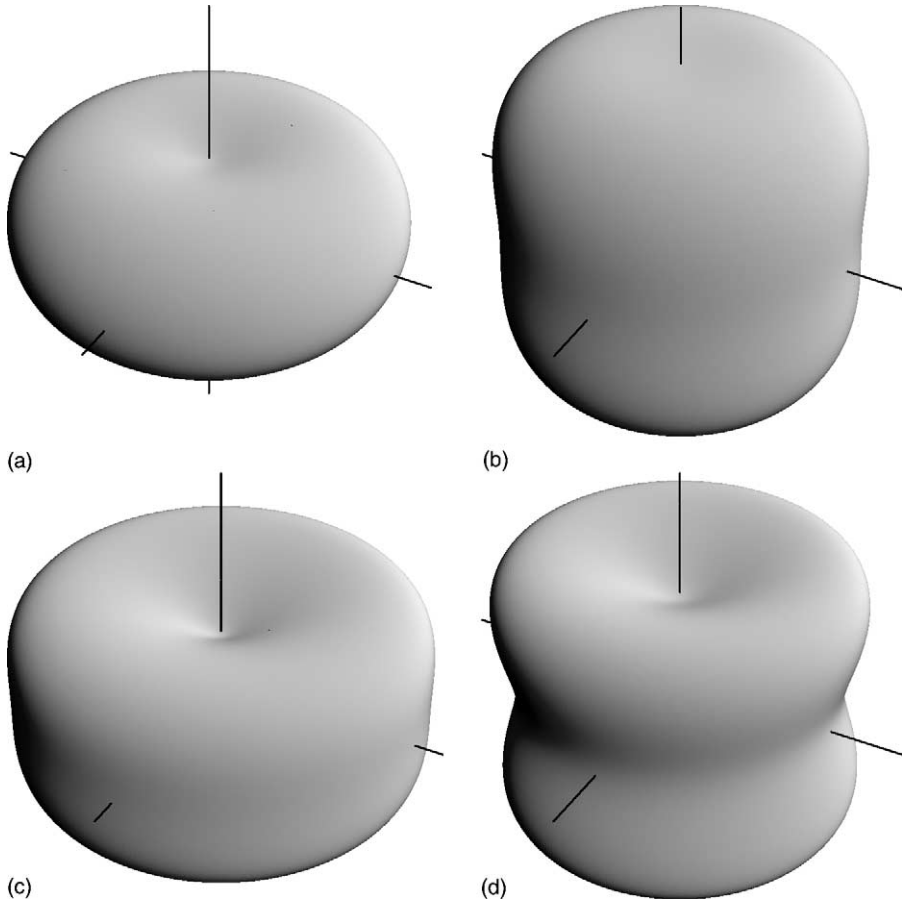

Fig. 9. Hexagonal system: plots of $E(\boldsymbol{n})$ for case 2, characterized by $\beta_2 > \alpha_2$ and $\alpha_2 < 0$. (a) $Cd_{1-x}Zn_x$ alloy ($x = 0.018\%$ content of Zn). $\alpha_2 = -22.72$, $\beta_2 = -7.35$ $(TPa)^{-1}$, $A_{hex} = 0.348$, $B_{hex} = 0.853$. The compliance coefficients (units in $(TPa)^{-1}$) are: $s_{11} = 12.55$, $s_{33} = 35.62$, $s_{44} = 49.78$, $s_{12} = -0.662$, $s_{13} = -9.76$. Young's modulus: $E_{min} = 0.028$ GPa and $E_{max} = 0.080$ GPa. (b) BN_3 (boron nitride): $\alpha_2 = -0.09$ and $\beta_2 = 4.84$ $(TPa)^{-1}$. $A_{hex} = 0.988$ and $B_{hex} = 1.314$. $s_{11} = 7.26$, $s_{33} = 7.35$, $s_{44} = 15.4$, $s_{12} = -3.98$ and $s_{13} = -2.86$ (in $(TPa)^{-1}$). $E_{min} = 0.136$ GPa and $E_{max} = 0.165$ GPa. (c) Zn (zinc): $\alpha_2 = -19.48$ and $\beta_2 = 5.14$ $(TPa)^{-1}$. $A_{hex} = 0.297$ and $B_{hex} = 1.203$. $s_{11} = 8.22$, $s_{33} = 27.7$, $s_{44} = 25.3$, $s_{12} = 0.60$ and $s_{13} = -7.0$ (in $(TPa)^{-1}$). $E_{min} = 0.036$ GPa and $E_{max} = 0.126$ GPa. (d) TiB_2 (titanium boride): $\alpha_2 = -1.36$ and $\beta_2 = 3.46$ $(TPa)^{-1}$. $A_{hex} = 0.655$ and $B_{hex} = 1.865$. $s_{11} = 2.58$, $s_{33} = 3.94$, $s_{44} = 4.00$, $s_{12} = -0.99$ and $s_{13} = -1.15$ (in $(TPa)^{-1}$). $E_{min} = 0.254$ GPa and $E_{max} = 0.510$ GPa. (Elastic coefficients taken from Landolt and Börnstein, 1992.)

Cd, Zn and apatite, among other materials, are representative of this class; the spherical polar diagram for some other real material belonging to this class are shown in Fig. 9.

3. $\beta_2 < \alpha_2$ and $\alpha_2 \geqslant 0$, i.e., $B' > (1 + A')/2$ and $0 < A' \leqslant 1$.

β_2	B'	E_{max}	θ_{max}	E_{med}	θ_{med}	E_{min}	θ_{min}
$0 < \beta_2 < \alpha_2$	$(1 + A')/2 < B' < 1$	E_1	θ_1	–	–	E_2	θ_2
$\beta_2 \leqslant 0$	$B' \geqslant 1$	E_1	θ_1	E_2	θ_2	$E_\star$	$\theta_\star$

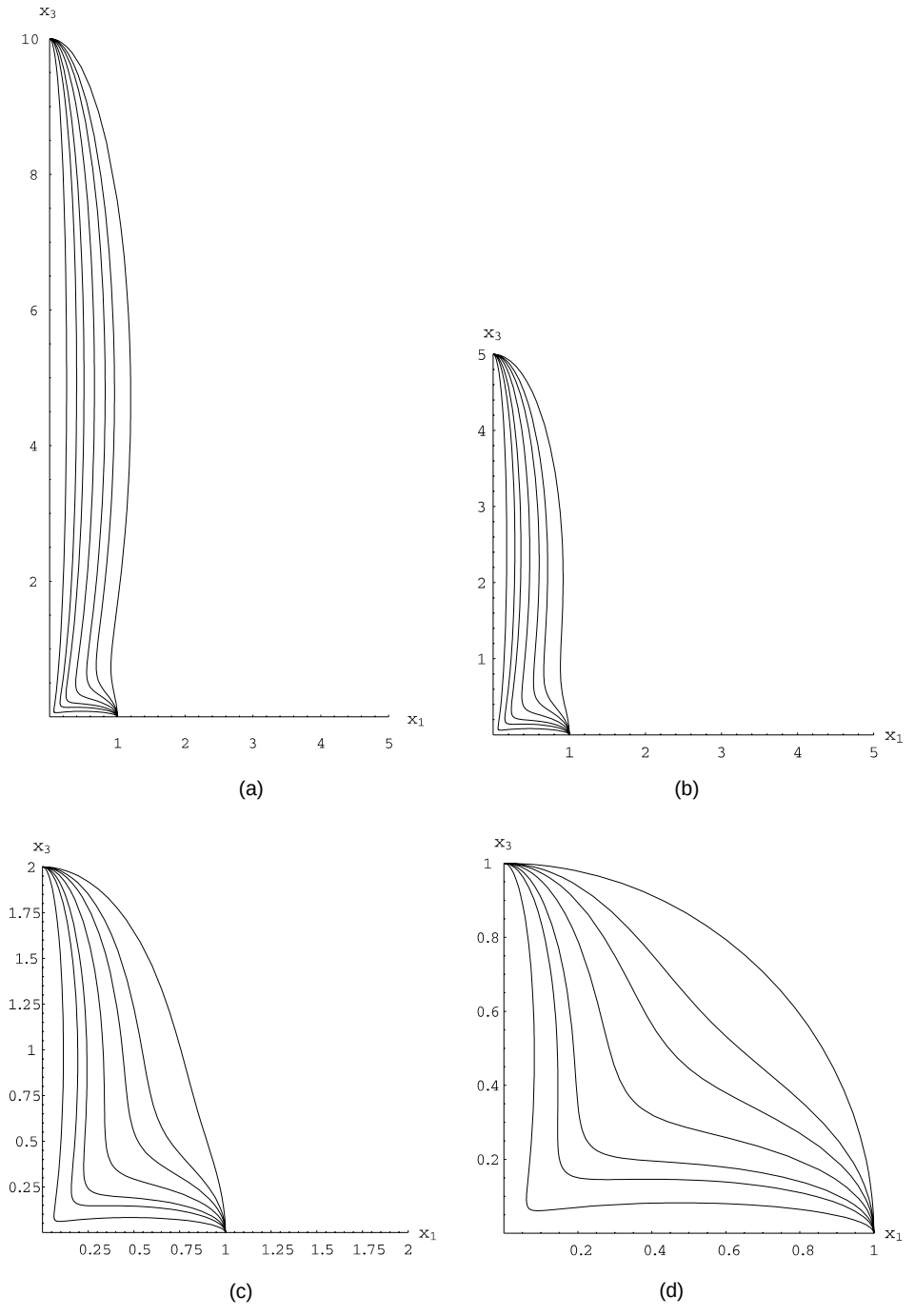

Fig. 10. Hexagonal system, case 3, characterized by $\beta_2 < \alpha_2$ and $\alpha_2 \geqslant 0$. Meridian sections of function $E(\boldsymbol{n})$ expressed in dimensionless variables A' and B' for $0 \leqslant \theta \leqslant \pi/2$. (a) $A' = 0.10$; $B' = 1.00$, 1.50, 2.00, 3.00, 5.00, 8.00, 20.00 (from outer to inner). (b) $A' = 0.20$; $B' = 1.00$, 1.50, 2.00, 3.00, 5.00, 8.00, 20.00 (from outer to inner). (c) $A' = 0.50$; $B' = 1.00$, 1.50, 2.00, 3.00, 5.00, 7.50, 20.00 (from outer to inner). (d) $A' = 1.00$; $B' = 1.01$, 1.50, 2.00, 3.00, 5.00, 7.50, 20.00 (from outer to inner).

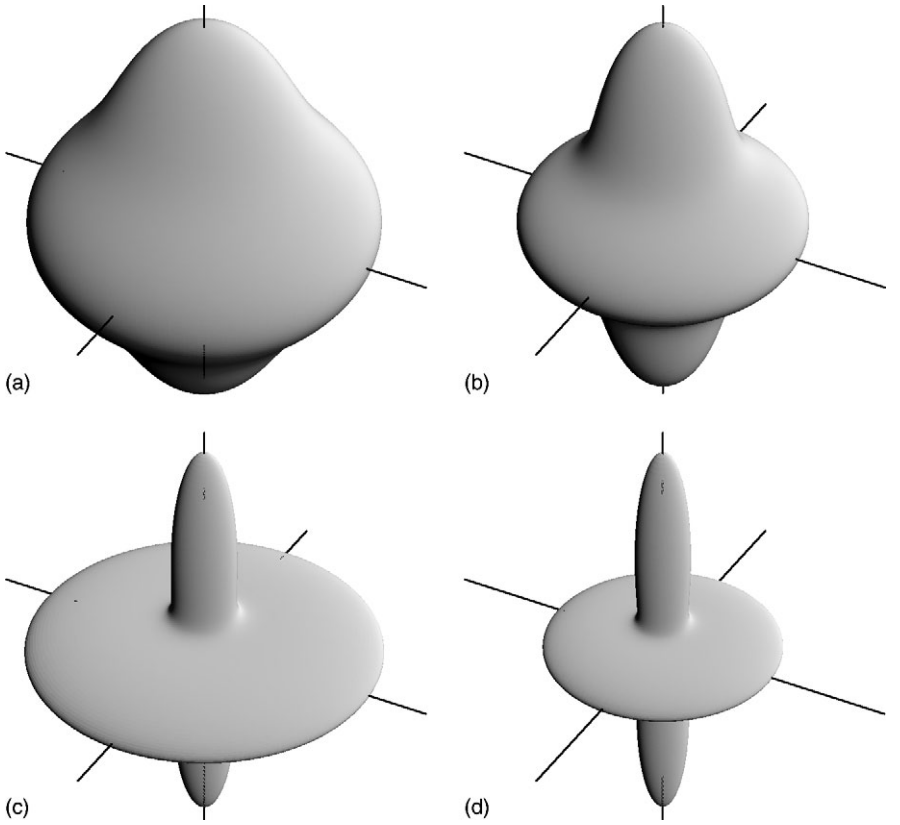

Fig. 11. Hexagonal system: plots of $E(\boldsymbol{n})$ for case 3, characterized by $\beta_2 < \alpha_2$ and $\alpha_2 \geqslant 0$. (a) Mendenhall Glacier ice (at 270 K). $\alpha_2 = 19.11$, $\beta_2 = -79.20$ $(\text{TPa})^{-1}$, $A_{\text{hex}} = 1.225$, $B_{\text{hex}} = 0.763$. The compliance coefficients (units in $(\text{TPa})^{-1}$) are: $s_{11} = 104.2$, $s_{33} = 85.09$, $s_{44} = 334.0$, $s_{12} = -43.3$, $s_{13} = -23.2$. Young's modulus: $E_{\min} = 0.006$ GPa and $E_{\max} = 0.012$ GPa. (b) CeF_3 (cesium fluoride): $\alpha_2 = 2.50$ and $\beta_2 = -11.48$ $(\text{TPa})^{-1}$. $A_{\text{hex}} = 1.486$ and $B_{\text{hex}} = 0.607$. $s_{11} = 7.64$, $s_{33} = 5.14$, $s_{44} = 29.2$, $s_{12} = -3.3$ and $s_{13} = -1.22$ (in $(\text{TPa})^{-1}$). $E_{\min} = 0.059$ GPa and $E_{\max} = 0.195$ GPa. (c) $RbMnCl_3$ (rubidium manganese chloride): $\alpha_2 = 4.90$ and $\beta_2 = -287.0$ $(\text{TPa})^{-1}$. $A_{\text{hex}} = 1.209$ and $B_{\text{hex}} = 0.196$. $s_{11} = 28.3$, $s_{33} = 23.4$, $s_{44} = 357.0$, $s_{12} = -9.0$ and $s_{13} = -6.7$ (in $(\text{TPa})^{-1}$). $E_{\min} = 0.003$ GPa and $E_{\max} = 0.043$ GPa. (d) $RbNiCl_3$ (rubidium nickel chloride): $\alpha_2 = 15.90$ and $\beta_2 = -309.60$ $(\text{TPa})^{-1}$. $A_{\text{hex}} = 1.807$ and $B_{\text{hex}} = 0.226$. $s_{11} = 35.6$, $s_{33} = 19.7$, $s_{44} = 400.0$, $s_{12} = -4.1$ and $s_{13} = -9.6$ (in $(\text{TPa})^{-1}$). $E_{\min} = 0.003$ GPa and $E_{\max} = 0.051$ GPa. (Elastic coefficients taken from Landolt and Börnstein, 1992.)

Meaningful examples of meridian sections for this class (restricted to the circumstance where all three stationary points exist in the range $0 \leqslant \theta \leqslant \pi/2$) are depicted in Fig. 10.

Representative elements of this class are, among others, Co, Ti, Hf, Y, Mg and ice; the spherical polar diagram for some other real material belonging to this class are shown in Fig. 11.

4. $\beta_2 < \alpha_2$ and $\alpha_2 < 0$ i.e., $B' > (1 + A')/2$ and $A' > 1$.

β_2	B'	$E_{\max}$	$\theta_{\max}$	E_{med}	θ_{med}	$E_{\min}$	$\theta_{\min}$
$2\alpha_2 < \beta_2 < \alpha_2$	$(1 + A')/2 < B' < A'$	E_2	θ_2	–	–	E_1	θ_1
$\beta_2 \leqslant 2\alpha_2$	$B' \geqslant A'$	E_2	θ_2	E_1	θ_1	$E_\star$	$\theta_\star$

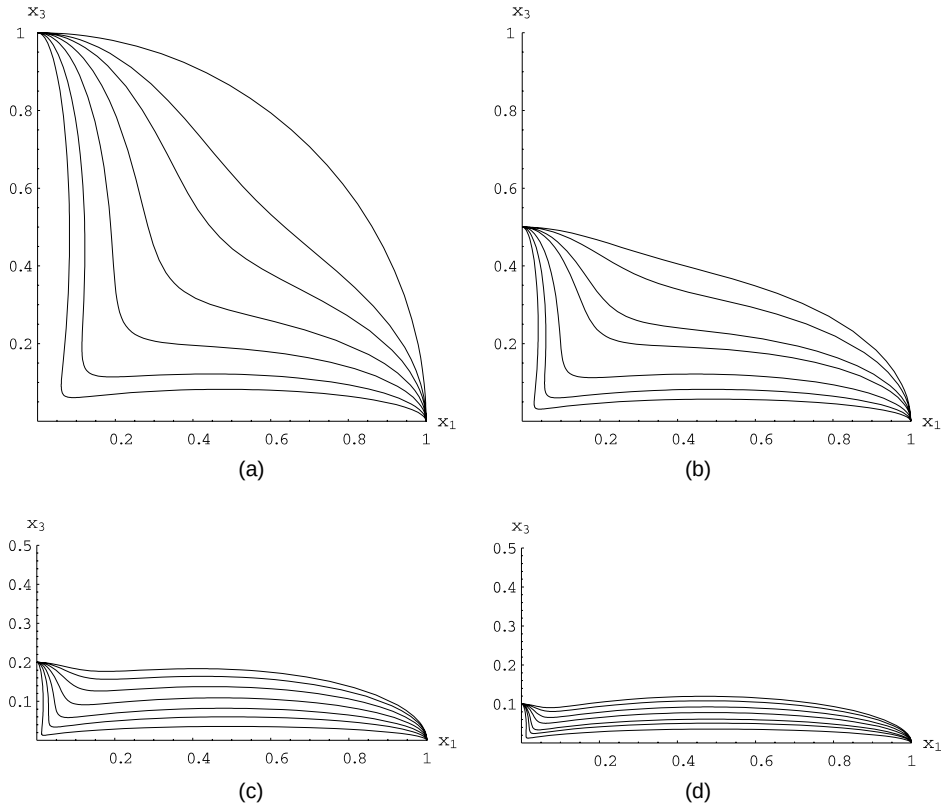

Fig. 12. Hexagonal system, case 4, characterized by $\beta_2 < \alpha_2$ and $\alpha_2 < 0$. Meridian sections of function $E(\boldsymbol{n})$ expressed in dimensionless variables A' and B' for $0 \leqslant \theta \leqslant \pi/2$. (a) $A' = 1.00$; $B' = 1.01$, 1.50, 2.00, 3.00, 5.00, 10.00, 20.00 (from outer to inner). (b) $A' = 2.00$; $B' = 2.00$, 2.50, 3.75, 5.00, 10.00, 20.00, 40.00 (from outer to inner). (c) $A' = 5.00$; $B' = 5.00$, 6.00, 8.00, 12.00, 20.00, 36.00, 100.00 (from outer to inner). (d) $A' = 10.00$; $B' = 10.00$, 12.00, 16.00, 22.00, 35.00, 50.00, 100.00 (from outer to inner).

Meaningful examples of meridian sections for this class (restricted to the circumstance where all three stationary points exist in the range $0 \leqslant \theta \leqslant \pi/2$) are depicted in Fig. 12.

Representative elements of this class are, among others, β-quartz, biotite and muscovite; the spherical polar diagram for some other real material belonging to this class are shown in Fig. 13.

5. Closure

For cubic and transversely isotropic solids the directions along which the Young's modulus attains stationary values have been provided. In particular, two different mechanical behaviors for the cubic symmetry and four different classes for transverse isotropy have been outlined. Such categories of behavior can be completely described through suitably defined material parameters, depending on the elastic compliances.

It has also been shown that all these classes occur in real materials, and a rather wide selection of the corresponding surfaces, showing in spherical polar diagrams the directional dependence of $E(\boldsymbol{n})$, has been given as well.

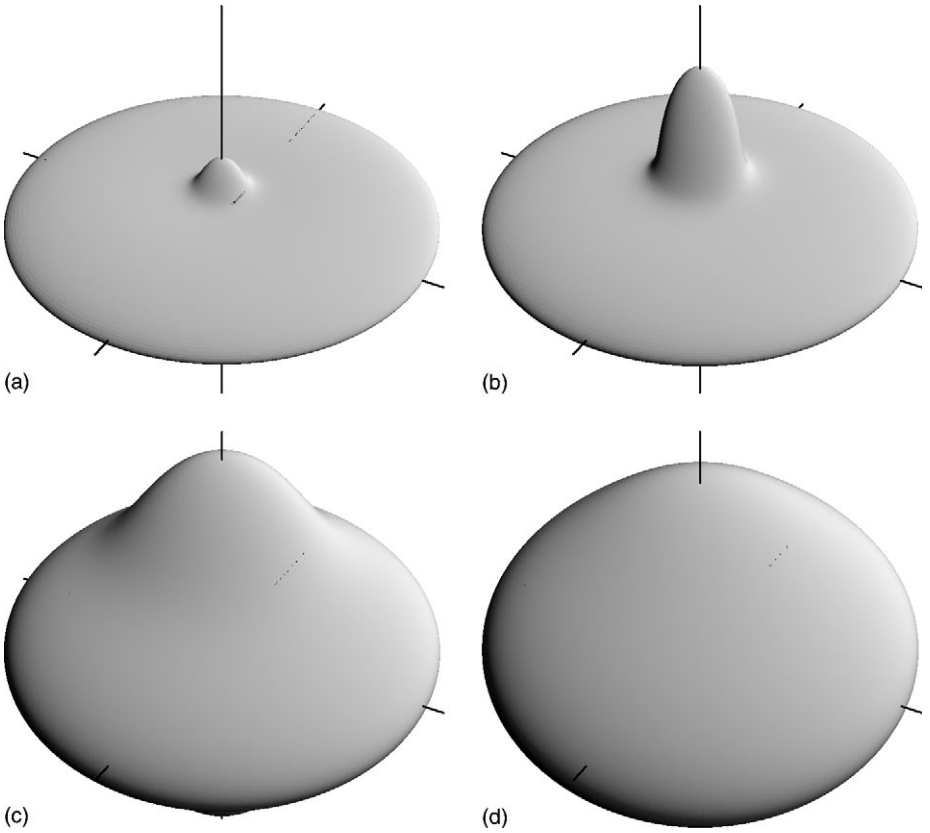

Fig. 13. Hexagonal system: plots of $E(\boldsymbol{n})$ for case 4, characterized by $\beta_2 < \alpha_2$ and $\alpha_2 < 0$. (a) $KAl_2Si_3AlO_{10}(OH,F)_2$ (muscovite). $\alpha_2 = -12.90$, $\beta_2 = -67.50$ $(TPa)^{-1}$, $A_{hex} = 0.317$, $B_{hex} = 0.176$. The compliance coefficients (units in $(TPa)^{-1}$) are: $s_{11} = 6.0$, $s_{33} = 18.9$, $s_{44} = 81.9$, $s_{12} = -1.3$, $s_{13} = -1.2$. Young's modulus: $E_{min} = 0.016$ GPa and $E_{max} = 0.167$ GPa. (b) InN (indium nitride): $\alpha_2 = -2.50$ and $\beta_2 = -71.80$ $(TPa)^{-1}$. $A_{hex} = 0.793$ and $B_{hex} = 0.289$. $s_{11} = 9.6$, $s_{33} = 12.1$, $s_{44} = 101.0$, $s_{12} = -2.1$ and $s_{13} = -5.0$ (in $(TPa)^{-1}$). $E_{min} = 0.026$ GPa and $E_{max} = 0.104$ GPa. (c) $Na_6Ca(AlSiO_4)_6CO_3 \cdot nH_2O$ (cancrinite): $\alpha_2 = -0.40$ and $\beta_2 = -15.60$ $(TPa)^{-1}$. $A_{hex} = 0.967$ and $B_{hex} = 0.625$. $s_{11} = 11.9$, $s_{33} = 12.3$, $s_{44} = 41.6$, $s_{12} = -5.37$ and $s_{13} = -1.1$ (in $(TPa)^{-1}$). $E_{min} = 0.063$ GPa and $E_{max} = 0.084$ GPa. (d) Cd–Mg (cadmium–magnesium alloy): $\alpha_2 = -4.30$ and $\beta_2 = -14.30$ $(TPa)^{-1}$. $A_{hex} = 0.849$ and $B_{hex} = 0.810$. $s_{11} = 24.2$, $s_{33} = 28.5$, $s_{44} = 75.1$, $s_{12} = -12.0$ and $s_{13} = -6.2$ (in $(TPa)^{-1}$). $E_{min} = 0.022$ GPa and $E_{max} = 0.041$ GPa. (Elastic coefficients taken from Landolt and Börnstein, 1992.)

The solution procedure here adopted can be straightforwardly applied also to weaker symmetry classes, even though an increased number of independent material constants would lead to more involved computations. Conversely, wider and more branched sets of behavioral sub-classes are likely to be generated. Moreover, the directional dependence of other significant moduli (e.g. shear modulus and Poisson's ratio) could be also of interest.

Acknowledgements

The financial support of both University of Trento (Italy) and MURST (the Italian Ministry for University and Scientific and Technological Research) is gratefully acknowledged. The authors feel also indebted to the reviewers for their hints in improving the manuscript.

References

Boulanger, P., Hayes, M., 1995. On Young's modulus for anisotropic media. ASME Journal of Applied Mechanics 62, 819–820.

Chadwick, P., Vianello, M., Cowin, S.C., 2001. A new proof that the number of linear elastic symmetries is eight. Journal of the Mechanics and Physics of Solids 49, 2471–2492.

Coleman, B.D., Noll, W., 1964. Material symmetry and thermostatic inequalities in finite elastic deformations. Archive for Rational Mechanics and Analysis 15, 87–111.

Edington, J.W., 1974. Interpretation of transmission electron micrographs. In: Monographs in Practical Electron Microscopy in Material Science. Philips Technical Library, Eindhoven.

Fedorov, F.I., 1968. Theory of Elastic Waves in Crystals. Plenum Press, New York.

Forte, S., Vianello, M., 1996. Symmetry classes for elasticity tensor. Journal of Elasticity 43, 81–108.

Goens, E., 1933. Über eine verbesserte apparatur zur statischen bestimmung des drillungsmodulus von kristallstäben und ihre anwendung auf zink-einkristalle. Annalen der Physik 5 (16), 793–809.

Gurtin, M.E., 1972. The linear theory of elasticity. Handbuch der Physik, Vol. VIa/2. Springer-Verlag, Berlin, pp. 1–295.

Hayes, M., Shuvalov, A., 1988. On the extreme values of Young's modulus, the shear modulus and Poisson's ratio for cubic materials. ASME Journal of Applied Mechanics 65, 786–787.

Hearmon, R.F.S., 1961. An Introduction to Applied Anisotropic Elasticity. Oxford University Press, Oxford.

Huntington, H.B., 1958. The elastic constants of crystals. Solid State Physics 7, 213–351.

Huo, Y.Z., Del Piero, G., 1991. On the completeness of the crystallographic symmetries in the description of the symmetries of the elastic tensor. Journal of Elasticity 25, 203–246.

Kelly, A., Groves, G.W., Kidd, P., 2000. Crystallography and Crystal Defects. John Wiley and Sons, Chichester.

Kocks, U.F., Tomé, C.N., Wenk, H.-R., 2000. Texture and Anisotropy. Preferred Orientations in Polycrystals and Their Effect on Material Properties. Cambridge University Press, Cambridge.

Landolt, H.H., Börnstein, R., 1992. Numerical Data and Functional Relationships in Science and Technology, III/29/a. Second and Higher Order Elastic Constants. Springer-Verlag, Berlin.

Lekhnitskii, S.G., 1981. Theory of Elasticity of an Anisotropic Body. Mir Publisher, Moscow.

Levy, M., Bass, H.E., Stern, R.R. (Eds.), 2001. Handbook of Elastic Properties of Solids, Liquids and Gases. Academic Press, San Diego.

Love, A.E.H., 1944. A Treatise on the Mathematical Theory of Elasticity, fourth ed. Dover, New York.

Mehrabadi, M.M., Cowin, S.C., 1990. Eigentensors of linear anisotropic elastic materials. The Quarterly Journal of Mechanics and Applied Mathematics 43 (1), 15–41.

Nadeau, J.C., Ferrari, M., 1998. Invariant tensor-to-matrix mappings for evaluation of tensorial expressions. Journal of Elasticity 52, 43–61.

Nadeau, J.C., Ferrari, M., 2001. On optimal zeroth-order bounds with application to Hashin–Shtrikman bounds and anisotropy parameters. International Journal of Solids and Structures 38, 7945–7965.

Nye, J.F., 1957. Physical Properties of Crystals. Their Representation by Tensors and Matrices. Oxford University Press, Oxford.

Ostrowska-Maciejewska, J., Rychlewski, J., 2001. Generalized proper states for anisotropic elastic materials. Archives of Mechanics 53 (4–5), 501–518.

Pedersen, P., 1989. On optimal orientation of orthotropic materials. Structural Optimization 1, 101–106.

Rychlewski, J., 1984. On Hooke's law. Prikladnaja Matematika i Mekhanika 48 (3), 420–435, Translated in: Journal of Applied Mathematics and Mechanics 48 (3), 303–314.

Rychlewski, J., 2000. A qualitative approach to Hooke's tensors. Part I, Archives of Mechanics 52 (4–5), 737–759.

Rychlewski, J., 2001. A qualitative approach to Hooke's tensors. Part II, Archives of Mechanics 53 (1), 45–63.

Schmid, E., Boas, W., 1935. Kristallplastizität mit Besonderer Berücksichtigung der Metalle. Julius Springer, Berlin.

Sirotin, Y., Chaskolkaïa, M., 1984. Fondements de la Physique des Cristaux. Éditions Mir, Moscow.

Smith, G.F., Rivlin, R.S., 1958. The strain energy function for anisotropic elastic materials. Transactions of the American Mathematical Society 88, 175–193.

Ting, T.C.T., 1996. Anisotropic Elasticity. Oxford University Press, New York.

Walpole, L.J., 1984. Fourth-rank tensors of the thirty-two crystal classes: multiplication tables. Proceedings of the Royal Society of London A 391, 149–179.

Wooster, W.A., 1949. A Text-book on Crystal Physics, second ed. Cambridge University Press, Cambridge.

Zener, C., 1955. Élasticité et Anélasticité des Métaux. Dunod, Paris.